SOLID GEOMETRY

HOBBS

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PREFACE

SOLID GEOMETRY

In passing from Plane Geometry to Solid Geometry many students find it hard to visualize in three dimensions a figure drawn on a plane surface. Hence it is necessary to give a beginner every possible assistance in forming space concepts. In this book the figures have been made as simple as possible in order that students not trained in drawing may be able readily to reproduce them.

Students should be encouraged to construct models corresponding to the figures given in the book. Much can be accomplished in this direction by the use of cardboard to represent planes and stout wire to represent lines. For the study of figures on a sphere, every class room should be provided with a globe having a blackboard surface. The use of colored crayons is helpful, especially when each separate plane or curved surface has a separate color given it.

The propositions corresponding to those of the National Syllabus are grouped as follows:

CLASS I. "Those of fundamental importance and basal character."

204, 242, 243, 244 – Cor., 249, 253, 254, 256, 304, 327, 332, 333, 334, 335, 341, 343.

CLASS II. "Those of considerable importance which are secondary only to the preceding ones."

232, 233, 236, 238, Ex. 1012, 247, 250, 252, 257, 258, 259, 260, 262 – Cor., 264, 267, 268, 269, 287, 289, 297, 299, *303, 321, 324, 328, 329.

Class III. "The student should be able to make a proof for anyone of them if allowed a reasonable interval for thought."

*203, †206, †207, †208, †209, 210, 211, 212, Ex. 958, 213, 214, 215, †216, †217, 218, 219, †220, 221, 222, 223, 224, 225, 226, 227, *228, *229, *230, 235 – Cor. II, 239, 245, 248 – Cor. II, 251, 261, *283, *292, 310, 312, 315, 316, 322, 342, 344.

Class IV. "The theorems may be used by the examiner with the understanding that they are to be regarded in examinations as of the nature of exercises."

231, 234, *237, 240, 241, 246, 265, *272, 284 – Cor. I, 293, 301, 317, 318, 319 – Cor. I, 323, 325, 330, 336, 337, 346 – Cor. II, 347.

(Propositions marked with an asterisk [*] are designated as "theorems for informal proof." For propositions marked with a dagger [†], only the latter part of the proof is required.)

For the volume of the frustum of a pyramid, for that of the frustum of a cone, and for that of the spherical segment, separate formulas have been developed in order that any teacher who so desires may treat each volume by itself. In the judgment of the author, however, it is more economical of time to compute these volumes by means of the prismatoidal formula.

CONTENTS

BOOK IX — THE SPHERE

REFERENCES TO PLANE GEOMETRY

SYMBOLS

$=$	is equal to	$\angle$, $\angle$s	angle, angles
$>$	is greater than	$\parallel$, $\parallel$s	parallel, parallels
$<$	is less than	$\perp$, $\perp$s	perpendicular, perpendiculars
$\equiv$	is congruent to	$\triangle$, $\triangle$s	triangle, triangles
$\sim$	is similar to	▱, ▱s	parallelogram, parallelograms
$\underline{\propto}$	is measured by	$\odot$, ⓢ	circle, circles
$\doteq$	approaches as a limit	$\frown$	arc

Similar figures are figures which have the same shape.

Equal figures are figures which have the same size.

Congruent figures are figures which have both the same shape and the same size. Equal figures which are not congruent are called *equivalent*.

The congruence of two figures may be tested by the method of superposition. If the figures coincide when one is placed upon the other, they are congruent.

In congruent or similar figures the corresponding parts are called *homologous*.

Homologous parts of congruent figures are equal.

Parallel lines are lines lying in the same plane which will never meet even if produced indefinitely.

A *polygon* is a plane closed figure bounded by straight lines. The sum of the bounding lines is called the *perimeter*.

A line joining two vertices of a polygon which are not consecutive is called a *diagonal.*

A polygon of three sides is called a *triangle;* one of four sides, a *quadrilateral;* one of five sides, a *pentagon;* one of six sides, a *hexagon;* one of eight sides, an *octagon;* one of ten sides, a *decagon;* one of twelve sides, a *dodecagon,* and so on.

An *isosceles* triangle has two sides equal: an *equilateral* triangle has all three sides equal.

A *parallelogram* is a quadrilateral which has its opposite side parallel.

A *rectangle* is a parallelogram whose angles are right angles. A *square* is a rectangle whose sides are equal.

Two polygons are *similar* when their homologous angles are equal and their homologous sides are proportional.

Homologous angles of similar polygons are equal.

Homologous sides of similar polygons are proportional.

A *circle* is a closed curve lying in a plane such that all points of the curve are equally distant from a fixed point in the plane called the *centre.*

When a variable, which changes according to some fixed law, can be made to have values such that the difference between the variable and a certain constant becomes and remains less than any assigned positive quantity, however small, the variable is said to *approach the constant as a limit,* and the constant is called the *limit* of the variable.

Ax. 1. *Quantities equal to the same quantity, or to equal quantities, are equal to each other.*

Ax. 2. *A quantity can be substituted for its equal in an equality or in an inequality.*

Ax. 3. *If three quantities are so related that the first is greater than the second, while the second is greater than the third, then the first is greater than the third.*

Ax. 4. *If equals are added to equals, the sums are equal.*

Ax. 5. *If equals are added to unequals, the sums are unequal in the same order.*

Ax. 6. *If unequals are added to unequals in the same order, the sums are unequal in the same order.*

Ax. 7. *If equals are subtracted from equals the remainders are equal.*

Ax. 8. *If equals are subtracted from unequals, the remainders are unequal in the same order.*

Ax. 9. *If unequals are subtracted from equals, the remainders are unequal in reverse order.*

Ax. 10. *If equals are multiplied by equals, the products are equal.*

Ax. 11. *If unequals are multiplied by equals, the products are unequal in the same order.*

Ax. 12. *If equals are divided by equals, the quotients are equal.*

Ax. 13. *If unequals are divided by equals, the quotients are unequal in the same order.*

Ax. 14. *Like powers or like roots of equals are equal.*

Ax. 15. *The whole is greater than any one of its parts.*

Ax. 16. *The whole is equal to the sum of all its parts.*

Post. 1. *Between two points one straight line can be drawn, and only one.*

Post. 2. *A straight line can be produced to any length.*

Post. 3. *A straight line is the shortest distance between two points.*

Post. 4. *Two straight lines can intersect in only one point.*

Post. 5. *A circle can be described about any given point as a centre with a radius of any given length.*

Post. 6. *A figure can be moved from one position to another without change of size or shape.*

Postulate of Parallels. *Two intersecting lines cannot both be parallel to the same line.*

Prop. 1. *At a given point in a given straight line one perpendicular to the line can be erected, and only one.*

Prop. 2. *All right angles are equal.*

Prop. 2, Cor. III. *The supplements of equal angles are equal.*

Prop. 3. *If one straight line meets another straight line not at its extremity, the sum of the two adjacent angles is equal to two right angles.*

Prop. 3, Cor. III. *The sum of all the successive angles formed about a point is equal to four right angles.*

Prop. 4. *If the sum of two adjacent angles is equal to two right angles, their exterior sides lie in the same straight line.*

Prop. 5. *If two straight lines intersect each other, the vertical angles are equal.*

Prop. 7. *The sum of two sides of a triangle is greater than the third side, and their difference is less than the third side.*

Prop. 8. *Two triangles are congruent when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

Prop. 8, Cor. *Two right triangles are congruent when the two legs of one are equal respectively to the two legs of the other.*

Prop. 9. *Two triangles are congruent when a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other.*

Prop. 10. *From a given point without a given straight line one perpendicular to the line can be drawn, and only one.*

Prop. 11. *Two right triangles are congruent when the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other.*

Prop. 12. *In an isosceles triangle the angles opposite the equal sides are equal.*

Prop. 12, Cor. *An equilateral triangle is also equiangular.*

Prop. 13. *The bisector of the vertical angle of an isosceles triangle bisects the base and is perpendicular to the base.*

Prop. 13, Cor. *In an isosceles triangle or an equilateral triangle the bisector of the vertical angle, the altitude, the median, and the perpendicular bisector of the base are one and the same straight line.*

Prop. 14. *If two angles of a triangle are equal, the sides opposite these angles are equal, and the triangle is isosceles.*

Prop. 14, Cor. *An equiangular triangle is also equilateral.*

Prop. 15. *Two triangles are congruent when the three sides of one are equal respectively to the three sides of the other.*

Prop. 16. *Two right triangles are congruent when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.*

Prop. 17. *If two angles of a triangle are unequal, the side opposite the greater angle is longer than the side opposite the less.*

Prop. 18. *If two sides of a triangle are unequal, the angle opposite the longer sides is greater than the angle opposite the shorter.*

Prop. 19. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.*

Prop. 20. *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*

Prop. 22. *If two oblique lines drawn from a point to a line meet the line at equal distances from the foot of the perpendicular drawn from the point to the line, they are equal.*

Prop. 22, Cor. I. *Every point in the perpendicular bisector of a line is equidistant from the extremities of the line.*

Prop. 23. *The perpendicular is the shortest line that can be drawn from a point to a line.*

Prop. 24. *If two oblique lines drawn from a point to a line meet the line at unequal distances from the foot of the perpendicular drawn from the point to the line, the more remote is the greater.*

Prop. 25. *The locus of a point equidistant from two given points is the perpendicular bisector of the line joining these points.*

Prop. 25, Cor. *Two points, each equally distant from the extremities of a line, determine the perpendicular bisector of the line.*

Prop. 26. *The locus of a point equidistant from two given intersecting lines is the pair of lines which bisect the angles formed by the given lines.*

Prop. 32. *Two lines in the same plane perpendicular to the same line are parallel.*

Prop. 33. *Through a given point without a given line one line can be drawn parallel to the given line, and only one.*

Prop. 34. *If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*

Prop. 35. *Two lines parallel to the same line are parallel to each other.*

Prop. 36. *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*

Prop. 37. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

Prop. 38. *If two parallel lines are cut by a transversal, the two interior angles on the same side of the transversal are supplementary.*

Prop. 39. *If two lines are cut by a transversal so as to make the alternate interior angles equal, the two lines are parallel.*

Prop. 40. *If two lines are cut by a transversal so as to make the corresponding angles equal, the two lines are parallel.*

Prop. 41. *If two lines are cut by a transversal so as to make the two interior angles on the same side of the transversal supplementary, the lines are parallel.*

Prop. 41, Cor. II. *Two lines respectively perpendicular to two intersecting lines cannot be parallel, and hence intersect.*

Prop. 43. *The sum of the three angles of a triangle is equal to two right angles.*

Prop. 43, Cor. I. *A triangle can have only one right angle, or one obtuse angle.*

Prop. 43, Cor. III. *If an acute angle of a right triangle is equal to an acute angle of another right triangle, the other acute angles are also equal.*

Prop. 43, Cor. IV. *In a right triangle, the two acute angles are complementary.*

Prop. 43, Cor. V. *Two right triangles are congruent when a leg and an acute angle of one are equal respectively to a leg and the homologous acute angle of the other.*

Prop. 44, Cor. *An exterior angle of a triangle is greater than either of the two opposite interior angles.*

Prop. 45. *Two angles whose sides are respectively parallel to each other are either equal or supplementary.*

Prop. 46. *Two angles whose sides are respectively perpendicular to each other are either equal or supplementary.*

Prop. 47. *If one acute angle of a right triangle is double the other, the shorter leg is equal to one half the hypotenuse.*

Prop. 47, Cor. *If the hypotenuse of a right triangle is double the shorter leg, the acute angles are 60° and 30°.*

Prop. 52. *Two parallelograms are congruent when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

Prop. 53. *The opposite sides of a parallelogram are equal.*

Prop. 53, Cor. I. *A diagonal of a parallelogram divides it into two congruent triangles.*

Prop. 53, Cor. II. *The segments of parallel lines included between parallel lines are equal.*

Prop. 54. *The opposite angles of a parallelogram are equal.*

Prop. 54, Cor. *If one angle of a parallelogram is a right angle, all the other angles are right angles, and the figure is a rectangle.*

Prop. 55. *The diagonals of a parallelogram bisect each other.*

Prop. 56. *The diagonals of a rectangle are equal.*

Prop. 57. *If two sides of a quadrilateral are equal and parallel, the other two sides are equal and parallel, and the figure is a parallelogram.*

Prop. 58. *If the opposite sides of a parallelogram are equal, the figure is a parallelogram.*

Prop. 61. *If three or more parallel lines intercept equal segments on one of two transversals, they intercept equal segments on the other transversal also.*

Prop. 61. Cor. *If a line which is parallel to one side of a triangle bisects a second side, it bisects the third side also.*

Prop. 63. *The line joining the mid-points of two sides of a triangle is parallel to the third side, and is equal to one half the third side.*

Prop. 64. *If a line which is parallel to the bases of a trapezoid bisects one leg, it bisects the other leg also.*

Prop. 65. *The median of a trapezoid is parallel to the parallel sides, and is equal to one half their sum.*

Prop. 66. *The bisectors of the three angles of a triangle are concurrent.*

Prop. 67. *The perpendicular bisectors of the three sides of a triangle are concurrent.*

Prop. 68. *The three altitudes of a triangle are concurrent.*

Prop. 69. *The three medians of a triangle are concurrent.*

Prop. 70. *The sum of the interior angles of a polygon is equal to as many right angles as twice the number of sides less four.*

Prop. 71. *If the sides of a polygon are produced so as to form an exterior angle at each vertex, the sum of the exterior angles is equal to four right angles.*

Prop. 72. *If two lines are divided proportionally, the ratio of either line to one of its segments is equal to the ratio of the other line to its corresponding segment.*

Prop. 73. *If a line is drawn through two sides of a triangle parallel to the third side, it divides the two sides proportionally.*

Prop. 76. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*

Prop. 79. *Two triangles are similar when they are mutually equiangular.*

Prop. 79, Cor. I. *Two triangles are similar when two angles of one are equal respectively to two angles of the other.*

Prop. 79, Cor. II. *Two right triangles are similar when an acute angle of one is equal to an acute angle of the other.*

Prop. 79, Cor. III. *If a line is drawn through two sides of a triangle parallel to the third side, the triangle thus formed is similar to the whole triangle.*

Prop. 80. *Two triangles are similar when an angle of one is equal to an angle of the other and the sides including these angles are proportional.*

Prop. 81. *Two triangles are similar when their homologous sides are proportional.*

Prop. 82. *Two triangles are similar when their sides are respectively parallel to each other.*

Prop. 83. *Two triangles are similar when their sides are respectively perpendicular to each other.*

Prop. 84. *If two triangles are similar, their holomogous altitudes are in the same ratio as any two homologous sides.*

Prop. 87. *If two polygons are similar, the diagonals drawn from homologous vertices divide them into the same number of similar triangles, similar each to each, and similarly placed.*

Prop. 88. *Two polygons are similar when they are composed of the same number of triangles, similar each to each, and similarly placed.*

Prop. 90. *The perimeters of two similar polygons are in the same ratio as any two homologous sides.*

Prop. 93. *In a right triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse, the perpendicular is a mean proportional between the segments of the hypotenuse, and either leg is a mean proportional between the whole hypotenuse and the adjacent segment.*

Prop. 94, Cor. *In a right triangle the perpendicular drawn from the vertex of the right angle to the hypotenuse divides the triangle into two triangles which are similar to the whole triangle, and also to each other.*

Prop. 96. *Any diameter bisects a circle.*

Prop. 98. *Through three points not in the same straight line one circle can be drawn, and only one.*

Prop. 100. *In the same circle, or in equal circles, equal central angles intercept equal arcs.*

Prop. 102. *In the same circle, or in equal circles, equal arcs subtend equal central angles.*

Prop. 104. *In the same circle, or in equal circles, equal chords subtend equal arcs.*

Prop. 105. *In the same circle, or in equal circles, equal arcs are subtended by equal chords.*

Prop. 108. *The diameter perpendicular to a chord bisects the chord and the arcs subtended by it.*

Prop. 108, Cor. *The perpendicular drawn from the centre of a circle to a chord bisects the chord.*

Prop. 109. *The diameter which bisects a chord is perpendicular to the chord, and bisects the subtended arcs.*

Prop. 112. *In the same circle, or in equal circles, equal chords are equally distant from the centre.*

Prop. 113. *In the same circle, or in equal circles, chords equally distant from the centre are equal.*

Prop. 116. *A line perpendicular to a radius at its extremity is a tangent to the circle.*

Prop. 117. *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

Prop. 120. *The two tangents drawn to a circle from an exterior point are equal.*

Prop. 124. *If two circles intersect each other, the line of centres is the perpendicular bisector of the common chord.*

Prop. 126. *In the same circle, or in equal circles, two central angles are in the same ratio as their intercepted arcs. — A central angle is measured by its intercepted arc.*

Prop. 127. *An inscribed angle is measured by one half its intercepted arc.*

Prop. 127, Cor. III. *An angle inscribed in a semicircle is a right angle.*

Prop. 145. *Parallelograms having equal bases and equal altitudes are equivalent.*

Prop. 146. *Triangles having equal bases and equal altitudes are equivalent.*

Prop. 146, Cor. II. *A triangle is equivalent to one half of a parallelogram having the same base and altitude.*

Prop. 149. *The area of a rectangle is equal to the product of its base and altitude.*

Prop. 149, Cor. *The area of a square is equal to the square of its side.*

Prop. 150. *The area of a parallelogram is equal to the product of its base and altitude.*

Prop. 151. *The area of a triangle is equal to one half the product of its base and altitude.*

Prop. 151, Cor. I. *Two triangles are to each other as the products of their bases and altitudes.*

Prop. 151, Cor. III. *Two triangles having equal bases are to each other as their altitudes.*

Prop. 152. *The area of a trapezoid is equal to the product of its altitude and one half the sum of its bases.*

Prop. 152, Cor. *The area of a trapezoid is equal to the product of its altitude and median.*

Prop. 153. *Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.*

Prop. 154. *Similar triangles are to each other as the squares of any two homologous sides.*

Prop. 155. *Similar polygons are to each other as the squares of any two homologous sides.*

Prop. 156. *The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the two legs.*

Prop. 158. *In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, diminished by twice the product of one of these sides and the projection of the other side upon it.*

Prop. 159. *In an obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of one of these sides and the projection of the other side upon it.*

Prop. 188. *Two regular polygons of the same number of sides are similar.*

Prop. 193. *If the number of sides of a regular polygon is increased indefinitely, the apothem of the polygon approaches the radius as its limit.*

Prop. 194. *If the number of sides of a regular inscribed or circumscribed polygon is increased indefinitely the perimeter of the polygon approaches the circumference of the circle as its limit.*

Prop. 195. *The circumferences of two circles are to each other as their radii.*

$$c = 2\pi r$$

Prop. 196. *If the number of sides of a regular inscribed or circumscribed polygon is increased indefinitely, the area of the polygon approaches the area of the circle as its limit.*

Prop. 198. *The area of a regular polygon is equal to one half the product of its perimeter and apothem.*

Prop. 199. *The area of a circle is equal to one half the product of its circumferences and radius.*

$$S = \pi r^2$$

SOLID GEOMETRY

Book VI

LINES AND PLANES IN SPACE

A *plane* is a surface such that if any two of its points be joined by a straight line, that line lies wholly in the surface.

The following principles follow at once from the definition of a plane:

(i) *A straight line which has two points in a plane lies wholly in the plane.*

(ii) *A straight line can intersect a plane in only one point.*

(iii) *Through any given straight line an infinite number of planes can be passed.*

Note. A plane is regarded as indefinite in extent, but only a portion of a plane can be shown in a diagram. It is customary to represent a plane by a quadrilateral, generally a parallelogram, lying in the plane.

A plane is said to be *determined* by given lines or points when one plane can be drawn which contains the given lines or points, and only one.

When a line meets a plane, the point in which it meets the plane is called the *foot* of the line.

Lines and points are said to be *coplanar* when they lie in the same plane.

The *intersection* of two surfaces is the locus of all the points common to the two surfaces.

Proposition 202 Theorem

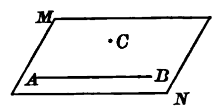
A plane is determined

(i) *by a straight line and a point without that line;*

(ii) *by three points not in the same straight line;*

(iii) *by two intersecting straight lines;*

(iv) *by two parallel lines.*



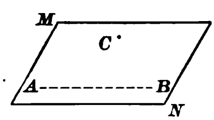
(i) **Hypothesis.** C is a point without the straight line AB.

Conclusion. A plane is determined by AB and C.

Proof. Let the plane MN, containing the line AB, revolve about AB as an axis until it contains the point C.

If MN is turned at all from this position, it no longer contains the point C.

Hence AB and C determine a plane.



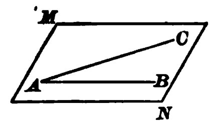
(ii) **Hypothesis.** A, B, and C are three points not in the same straight line.

Conclusion. A plane is determined by A, B, and C.

Proof. Draw AB.

The line AB and the point C determine a plane.

Hence the three points A, B, and C determine a plane.



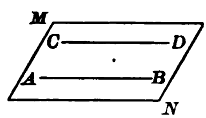
(iii) **Hypothesis.** AB and AC are two intersecting straight lines.

Conclusion. A plane is determined by AB and AC.

Proof. The line AB and the point C, any point in AC except A, determine a plane.

Since the line AC has the two points A and C in this plane, the line lies wholly in the plane.

Hence AB and AC determine a plane.



(iv) **Hypothesis.** AB and CD are two parallel lines.

Conclusion. A plane is determined by AB and CD.

Proof. Since AB and CD are parallel, they both lie in the same plane.

There is only one such plane, for AB and C, any point in CD, determine a plane.

Hence AB and CD determine a plane.

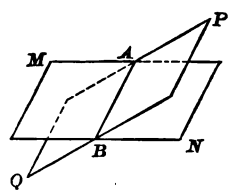
Q. E. D.

COR. *Two intersecting lines or two parallel lines are coplanar.*

REMARK. When the lines and points of a figure lie in the same plane, the propositions of Plane Geometry can be applied. In some cases (notably Prop. 1) it is necessary to add the words "in the same plane" in order to make the propositions applicable to Solid Geometry. Before making use of propositions of Plane Geometry, the student should satisfy himself that they are true in the figures under consideration.

Proposition 203 Theorem

If two planes intersect, their intersection is a straight line.



Hypothesis. MN and PQ are two intersecting planes.

Conclusion. The intersection of MN and PQ is a straight line.

Proof. Let A and B be two points common to the two planes.

Draw the straight line AB.

The straight line AB lies wholly in both planes. (?)

Moreover, no point without the straight line AB can lie in both planes. (?)

$\therefore$ AB is the locus of all the points common to both planes, and their intersection is a straight line.

Q.E.D.

REMARK. The figure of Proposition 203 illustrates a further use of dotted lines in addition to their use as construction lines. Whenever a part of a figure in space would be hidden from an observer by some other part of the figure, the lines of the hidden part are represented in a diagram by dotted lines.

SKEW LINES

Lines which are not coplanar are called *skew* lines. Accordingly, skew lines include all lines which do not intersect or are not parallel.

In Plane Geometry two intersecting lines are either perpendicular to each other or are oblique to each other. Likewise, in Solid Geometry two skew lines are either perpendicular to each other or are oblique to each other. If a line is drawn intersecting one of two skew lines parallel to the other, the two skew lines are perpendicular or oblique to each other, according as the two intersecting lines are perpendicular or oblique to each other.

RELATIONS OF LINES AND PLANES

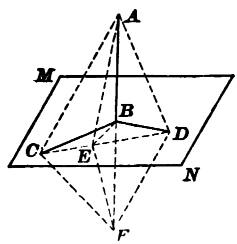
A line is said to be *perpendicular to a plane* when it is perpendicular to every line in the plane drawn through its foot; at the same time the plane is said to be perpendicular to the line.

A line is said to be *parallel to a plane* when the line and the plane will never meet even if produced indefinitely; at the same time the plane is said to be parallel to the line. Two planes are said to be *parallel* when they will never meet even if produced indefinitely.

A line which is neither perpendicular nor parallel to a plane is said to be *oblique* to the plane.

Proposition 204 Theorem

If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane determined by these lines.



Hypothesis. The line AB is $\perp$ to the lines BC and BD at their point of intersection, and MN is the plane determined by BC and BD.

Conclusion. AB is $\perp$ to MN.

Proof. Through B draw BE, any other line in MN. Also draw CD in MN cutting BC, BE, and BD at C, E, and D respectively.

Produce AB to F, making $BF = AB$, and draw AC, AD, AE, FC, FD, and FE.

$AC = FC$, and $AD = FD$. (?)

$\triangle ACD \equiv \triangle FCD$. (?)

$\therefore \angle ACE = \angle FCE$. (?)

$\triangle ACE \equiv \triangle FCE$. (?)

$\therefore AE = FE$. (?)

Then BE is $\perp$ to AF. (?)

That is, AB is $\perp$ to BE. (?)

In like manner it may be proved that AB is $\perp$ to every line in MN drawn through B.

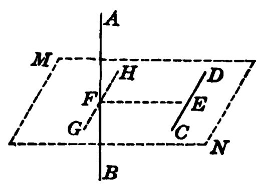
$\therefore AB$ is $\perp$ to MN. (?)

Q.E.D.

COR. *If two intersecting lines are perpendicular to a third line at the same point, their plane is perpendicular to that line.*

Proposition 205 Theorem

If two skew lines are perpendicular to each other, a plane can be drawn through either line perpendicular to the other.



Hypothesis. AB and CD are two skew lines $\perp$ to each other.

Conclusion. A plane can be drawn through CD $\perp$ to AB.

Proof. From E, any point in CD, draw $EF \perp$ to AB.

Through F draw $GH \parallel$ to CD.

Let MN be the plane determined by EF and GH.

AB is $\perp$ to GH. (?)

Then AB is $\perp$ to MN, a plane drawn through CD. (?)

Q.E.D.

COR. *If a line is perpendicular to a plane, it is perpendicular to any line in the plane.*

Ex. 952. When is it possible to pass a plane through one of two lines perpendicular to the other? Give reasons for your answer.

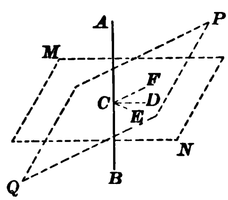
Ex. 953. Show how to construct a line passing through a given point and intersecting each of two given skew lines. When is the construction impossible?

Ex. 954. Show how to draw a line intersecting three given non-intersecting lines. How many such lines can be drawn?

Ex. 955. Construct in a given plane P a line x through a given point A in the plane such that x shall be perpendicular to a line which does not lie in P or pass through A.

Proposition 206 Theorem

Through a given point in a given line one plane perpendicular to the line can be drawn, and only one.



Hypothesis. C is a point in the line AB.

Conclusion. Through C one plane $\perp$ to AB can be drawn, and only one.

Proof. At C erect CD and CE $\perp$ to AB in two different planes, and let MN be the plane determined by CD and CE.

Then MN is $\perp$ to AB. (?)

If possible, suppose that PQ is another plane $\perp$ to AB at C, and let the plane determined by AB and CD intersect PQ in the line CF.

Then AB is $\perp$ to CF. (?)

But this is impossible. (?)

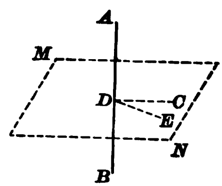
∴ PQ cannot be ⊥ to AB, and there can be only one plane ⊥ to AB at the point C.

Q. E. D.

Discussion. In Solid Geometry there are several theorems of the same nature as Prop. 206. These theorems show the possibility of constructing lines and planes under certain given conditions. No attempt is made to show how to perform these constructions, as this would involve the use of more than one plane. In order to make an accurate representation of such a problem in a plane figure, the student must make a study of Descriptive Geometry.

Proposition 207 Theorem

Through a given point without a given line one plane perpendicular to the line can be drawn, and only one.



Hypothesis. C is a point without the line AB.

Conclusion. Through C one plane ⊥ to AB can be drawn, and only one.

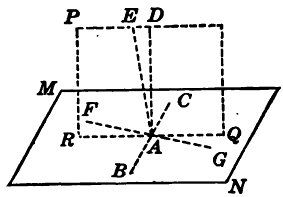
Proof. Draw CD ⊥ to AB, and at D erect DE ⊥ to AB in any plane other than the plane determined by AB and DC. Let MN be the plane determined by DC and DE.

Then MN is ⊥ to AB. (?)

[To prove that there can be only one ⊥, use a method similar to that used in Prop. 206.]

Proposition 208 Theorem

At a given point in a given plane one perpendicular to the plane can be erected, and only one.



Hypothesis. A is a point in the plane MN.

Conclusion. At A one $\perp$ to MN can be erected, and only one.

Proof. Through A draw any line BC in MN.

Through A pass the plane PQ $\perp$ to BC, intersecting MN in RQ.

At A erect AD $\perp$ to RQ in the plane PQ.

BC is $\perp$ to AD. (?)

$\therefore$ AD is $\perp$ to MN. (?)

If possible, suppose that AE is another $\perp$ to MN erected at A, and let the plane determined by AD and AE intersect MN in FG.

Then AD and AE are both $\perp$ to FG. (?)

But this is impossible. (?)

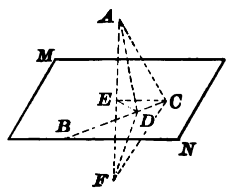
$\therefore$ AE cannot be $\perp$ to MN, and AD is the only $\perp$ to MN that can be erected at A.

Q. E. D.

Ex. 956. Lines a and b are both perpendicular to line c. Are a and b necessarily parallel? Do a and b necessarily lie in the same plane? Explain your answers.

Proposition 209 Theorem

From a given point without a given plane one perpendicular to the plane can be drawn, and only one.



Hypothesis. A is a point without the plane MN.

Conclusion. From A one $\perp$ to MN can be drawn, and only one.

Proof. Draw any line BC in the plane MN, and from A draw $AD \perp$ to BC. At D erect $DE \perp$ to BC in the plane MN, and from A draw $AE \perp$ to DE.

Produce AE to F, making $EF = AE$, and draw DF. From C, any point in BC, draw CA, CE, and CF.

BC is $\perp$ to the plane determined by AD and DE. (?)

$\therefore BC$ is $\perp$ to DF. (?)

$\triangle ADC \equiv \triangle FDC$. (?)

$\therefore AC = FC$. (?)

Then EC is $\perp$ to AF. (?)

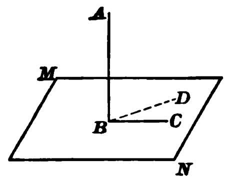
$\therefore AE$ is $\perp$ to the plane MN. (?)

[To prove that there can be only one $\perp$, use a method similar to that used in Prop. 208.]

Ex. 957. Is it true that if two lines are perpendicular to each other, any plane passed through one of the lines is perpendicular to the other? Explain your answer.

Proposition 210 Theorem

Every perpendicular to a line at a given point lies in the plane perpendicular to the line at the given point.



Hypothesis. BC is any line $\perp$ to the line AB at B, and the plane MN is $\perp$ to AB at B.

Conclusion. BC lies in MN.

Proof. Let the plane determined by AB and BC intersect MN in the line BD.

Then BD is $\perp$ to AB. (?)

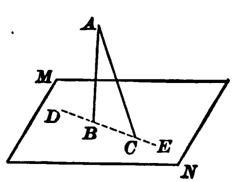
Hence BD and BC coincide. (?)

$\therefore$ BC lies in MN. (?)

Q. E. D.

Proposition 211 Theorem

The perpendicular is the shortest line that can be drawn from a point to a plane.



Hypothesis. AB is the $\perp$ drawn from A to the plane MN, and AC is any other line drawn from A to MN.

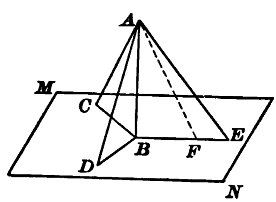
Conclusion. $AB < AC$.

HINT. Let the plane determined by AB and AC intersect MN in the line DE.

DEFINITION. The *distance* from a point to a plane is the length of the perpendicular drawn from the point to the plane.

Proposition 212 Theorem

If two oblique lines drawn from a point to a plane meet the plane at equal distances from the foot of the perpendicular drawn from the point to the plane, they are equal; and if two oblique lines meet the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.



Hypothesis. AB is $\perp$ to the plane MN, and the oblique lines AC, AD, and AE are so drawn that $BC = BD$, and $BE > BC$.

Conclusion. $AC = AD$, and $AE > AC$.

First Part. Consult Prop. 8, Cor.

Second Part. Take $BF = BC$, and draw AF. Consult Prop. 24.

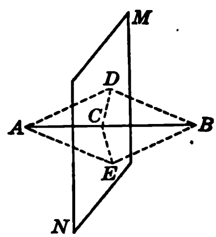
Ex. 958. State and prove the converse of Prop. 212. Use the indirect method.

Ex. 959. If from the foot of a perpendicular to a plane a line is drawn at right angles to any line in the plane, the line drawn from its intersection with the line in the plane to any point in the perpendicular is perpendicular to the line in the plane.

Ex. 960. If several planes have a common line of intersection, the perpendiculars to these planes erected at any point of this line all lie in one plane.

Proposition 213 Theorem

The locus of a point equidistant from the extremities of a line is the plane perpendicular to the line at its mid-point.



Hypothesis. MN is the plane $\perp$ to the line AB at its mid-point.

Conclusion. MN is the locus of a point equidistant from A and B.

Proof. It is necessary to prove that:

(i) every point in MN is equidistant from A and B;

(ii) every point equidistant from A and B lies in MN.

Let D be any point in MN. Consult Prop. 22, Cor. I.

Let E be any point equidistant from A and B. Consult Prop. 25, Cor. and Prop. 210.

DEFINITION. The *locus of a line* is the figure which includes all the points of all the lines that satisfy a given condition (or conditions), and no others.

Ex. 961. Find the locus of a line which is perpendicular to a given line at a given point.

Ex. 962. Find the locus of a point in a plane at a given distance from a point without the plane.

Ex. 963. Find the locus of a point in space equidistant from all points in a circle.

Ex. 964. Find the locus of a point in space equidistant from three given points not in the same straight line.

Ex. 965. Find the locus of a point in a given plane which is equidistant from two given points without the plane.

Ex. 966. Find the locus of a point in space equidistant from three intersecting lines lying in the same plane.

Ex. 967. To find a point in a given line which shall be equidistant from two given points in space.

Ex. 968. To find a point in a given plane which shall be equidistant from three given points in space.

Ex. 969. To find a point in a given plane equidistant from all points in a circle not lying in the plane.

Ex. 970. To find a point at equal distances from four points not all in the same plane.

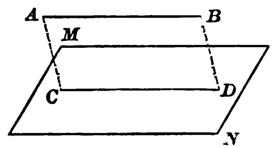
Proposition 214 Theorem

The intersections of two parallel planes with any third plane are parallel.

HINT. Prove that the lines of intersection are in the same plane and that they can never meet.

Proposition 215 Theorem

If a line without a plane is parallel to a line in the plane, it is parallel to the plane.



Hypothesis. AB, a line without the plane MN, is ∥ to CD, a line in MN.

Conclusion. AB is ∥ to MN.

Proof. If possible, suppose that AB is not ∥ to MN.

Then AB meets the plane MN at some point, and this point must lie in both the planes MN and AD; that is, it must lie in CD, the intersection of the two planes.

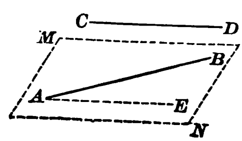
But AB can never meet CD. (?)

∴ AB is ∥ to MN. (?)

Q.E.D.

Proposition 216 Theorem

Through either of two skew lines one plane parallel to the other can be drawn, and only one.



Hypothesis. AB and CD are two skew lines.

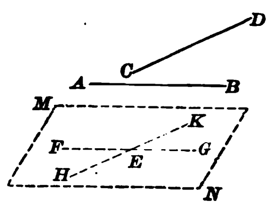
Conclusion. Through AB one plane ∥ to CD can be drawn, and only one.

Proof. Through A draw $AE \parallel$ to CD.

[To be completed by the student.]

Proposition 217 Theorem

Through a given point without two given non-parallel lines one plane can be drawn parallel to the given lines, and only one.



Hypothesis. E is a point without the non-parallel lines AB and CD.

Conclusion. Through E one plane $\parallel$ to AB and CD can be drawn, and only one.

Proof. Through E draw $FG \parallel$ to AB and $HK \parallel$ to CD.

[To be completed by the student.]

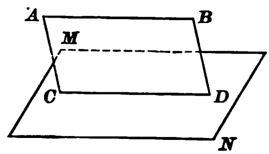
Ex. 971. Through a given point without two given intersecting planes one line can be drawn parallel to each of the planes, and only one.

Ex. 972. If a line and a plane are perpendicular to the same line, they are parallel.

Ex. 973. AB, BC, and CD are lines not all in one plane; prove that the plane determined by the mid-points of these three lines is parallel to both AC and BD.

Proposition 218 Theorem

If a line is parallel to a plane, the intersection of the plane with any plane passed through the line is parallel to the line.



Hypothesis. The line AB is ‖ to the plane MN, and CD is the intersection of MN with AD, a plane passing through AB.

Conclusion. CD is ‖ to AB.

[The proof is left to the student.]

NOTE. If the plane passing through AB does not intersect MN, the planes are parallel.

Ex. 974. Are lines which are parallel to the same plane parallel to each other? Illustrate by a figure.

Ex. 975. Is it true that if a plane is parallel to a second plane, every line in the first plane is parallel to the second plane? Illustrate by a figure.

Ex. 976. Is it true that if a line is parallel to one of two parallel planes, it is parallel to the other? Illustrate by a figure.

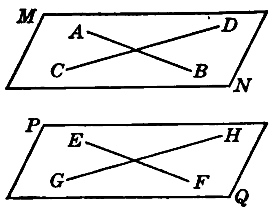
Ex. 977. Is it true that if a plane is parallel to one of two parallel lines, it is parallel to the other? Illustrate by a figure.

Ex. 978. Is it true that planes parallel to the same line are parallel to each other? Illustrate by a figure.

Ex. 979. If a line is parallel to a plane, a line parallel to the given line through any point of the plane lies in the plane.

Proposition 219 Theorem

If two intersecting lines in one plane are respectively parallel to two intersecting lines in another plane, the two planes are parallel.



Hypothesis. The lines AB and CD are respectively ∥ to the lines EF and GH. MN is the plane determined by AB and CD, and PQ is the plane determined by EF and GH.

Conclusion. MN is ∥ to PQ.

Proof. If possible, suppose that MN is not ∥ to PQ, and that the planes intersect.

AB and CD are both ∥ to PQ. (?)

Hence the intersection of MN and PQ is ∥ to both AB and CD. (?)

But this is impossible. (?)

∴ MN is ∥ to PQ. (?)

Q. E. D.

Proposition 220 Theorem

Through a given point without a given plane one plane can be drawn parallel to the given plane, and only one.

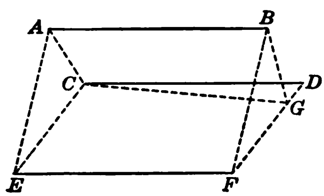
HINT. Draw two intersecting lines in the given plane; through the given point draw lines parallel to these lines.

Ex. 980. If two intersecting lines are both parallel to a plane, the plane determined by these lines is parallel to the plane.

Ex. 981. Find the locus of a line which passes through a given point and is parallel to a given plane.

Proposition 221 Theorem

Two lines parallel to the same line are parallel to each other.



Hypothesis. The lines AB and CD are both ∥ to the line EF.

Conclusion. AB is ∥ to CD.

Proof. Let the planes determined by AB and EF and by CD and EF be AF and CF respectively.

Pass a plane AG through AB and the point C, and let this plane intersect plane CF in the line CG.

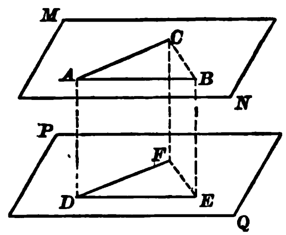
EF is ∥ to plane AG.	(?)
∴ EF is ∥ to CG.	(?)
But EF is ∥ to CD.	(?)
∴ CD and CG coincide.	(?)
Now AB is ∥ to plane CF.	(?)
∴ AB is ∥ to CG.	(?)
∴ AB is ∥ to CD.	(?)

Q. E. D.

Ex. 982. If a line is parallel to each of two intersecting planes, it is parallel to their line of intersection.

Proposition 222 Theorem

If two angles, not in the same plane, have their sides respectively parallel and extending in the same direction, they are equal.



Hypothesis. $\angle$s BAC and EDF are in the planes MN and PQ respectively. AB and AC are $\parallel$ to DE and DF respectively, and extend in the same direction.

Conclusion. $\angle BAC = \angle EDF$.

Proof. Take $AB = DE$ and $AC = DF$.

Draw BC, EF, AD, BE, and CF.

BE and AD are equal and $\parallel$, and CF and AD are equal and $\parallel$. (?)

$\therefore BE$ and CF are equal and $\parallel$. (?)

$\therefore BC = EF$. (?)

$\therefore \triangle ABC \equiv \triangle DEF$. (?)

$\therefore \angle BAC = \angle EDF$. (?)

Q. E. D.

NOTE. Prop. 45 is the corresponding theorem for angles in the same plane. The same conditions govern here in the cases of equal or supplementary angles.

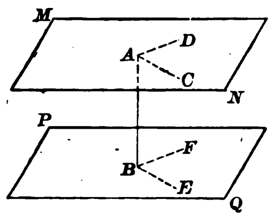
Proposition 223 Theorem

Two planes perpendicular to the same line are parallel.

Use the indirect method. Consult Prop. 207.

Proposition 224 Theorem

If a line is perpendicular to one of two parallel planes, it is perpendicular to the other also.



Hypothesis. The planes MN and PQ are ∥, and the line AB is ⊥ to MN.

Conclusion. AB is ⊥ to PQ.

Proof. Through B draw two lines BF and BE in the plane PQ. Let the planes determined by AB and BE and by AB and BF intersect MN in the lines AC and AD respectively.

AC is ∥ to BE, and AD is ∥ to BF. (?)

AB is ⊥ to AC and AD. (?)

∴ AB is ⊥ to BE and BF. (?)

∴ AB is ⊥ to PQ. (?)

Q.E.D.

Ex. 983. **The segments of parallel lines included between parallel planes are equal.**

Ex. 984. Two parallel planes are everywhere equally distant.

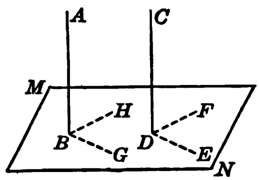
Ex. 985. Two planes parallel to the same plane are parallel.

Ex. 986. Find the locus of a point at a given distance from a given plane.

Ex. 987. Find the locus of a point equidistant from two given parallel planes.

Proposition 225 Theorem

If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.



Hypothesis. AB and CD are $\parallel$ lines, and AB is $\perp$ to the plane MN.

Conclusion. CD is $\perp$ to MN.

Proof. Through D draw two lines DE and DF in the plane MN.

Draw BG and BH in MN $\parallel$ to DE and DF respectively.

$\angle ABG = \angle CDE$ and $\angle ABH = \angle CDF$. (?)

$\angle$s ABG and ABH are rt. $\angle$s. (?)

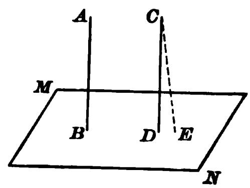
$\therefore$ $\angle$s CDE and CDF are rt. $\angle$s, and CD is $\perp$ to both DE and DF. (?)

$\therefore$ CD is $\perp$ to MN. (?)

Q. E. D.

Proposition 226 Theorem

Two lines perpendicular to the same plane are parallel.



Hypothesis. The lines AB and CD are both $\perp$ to the plane MN.

Conclusion. AB is $\parallel$ to CD.

HINT. From any point in CD draw CE $\parallel$ to AB, and prove that CD and CE coincide.

COR. *Two lines respectively perpendicular to two intersecting planes cannot be parallel.*

Ex. 988. If a line is bisected by a plane, its extremities are equally distant from the plane.

Ex. 989. If a plane is passed through a diagonal of a parallelogram, perpendiculars to this plane drawn from the extremities of the other diagonal are equal.

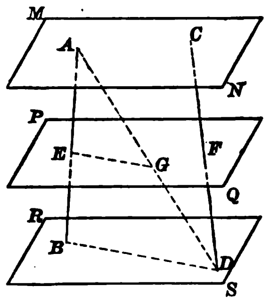
Ex. 990. If perpendiculars to a plane are drawn from the vertices of a parallelogram lying without the plane, the sum of these perpendiculars is equal to four times the perpendicular drawn to the plane from the point of intersection of the diagonals of the parallelogram.

Ex. 991. If two points lie on the same side of a plane and are equidistant from it, they determine a line parallel to the plane.

Ex. 992. If two points lie on opposite sides of a plane and are equidistant from it, the line joining them is bisected by the plane.

Proposition 227 Theorem

If two lines are cut by three parallel planes, the corresponding segments are proportional.



Hypothesis. The lines AB and CD are cut by the ∥ planes MN, PQ, and RS in the points A, E, and B and C, F, and D respectively.

Conclusion. $$\frac{AE}{EB} = \frac{CF}{FD}.$$

Proof. Draw AD, intersecting PQ at G.

Let the plane determined by AB and AD intersect PQ in the line EG and RS in the line BD.

Then EG is ∥ to BD. (?)

$$\therefore \frac{AE}{EB} = \frac{AG}{GD}. \quad (?)$$

In like manner it can be proved that $\frac{AG}{GD} = \frac{CF}{FD}$.

$$\therefore \frac{AE}{EB} = \frac{CF}{FD}. \quad (?)$$

Q. E. D.

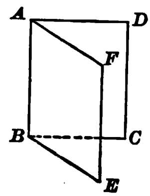
Cor. *If two lines drawn from the same point are cut by two or more parallel planes, the corresponding segments are proportional.*

Ex. 993. Given two parallel planes and a line perpendicular to one of them cutting them at A and B. A third plane parallel to the first two bisects AB; prove that it also bisects the line joining any point of the first plane to any point of the second.

DIHEDRAL ANGLES

A *dihedral angle* is the opening between two intersecting planes. The line of intersection is called the *edge* of the dihedral angle, and the two planes are called its *faces*.

A dihedral angle may be designated by four letters, two on the edge, and one other in each face, the letters on the edge being placed between the other two. For example, the planes AC and AE, intersecting in the line AB, form the dihedral angle $CABF$.



When one dihedral angle stands alone, it may be designated by two letters on the edge; thus, the dihedral angle in the figure above may be designated by AB.

The size of a dihedral angle is entirely independent of the extent of the faces. The best way to consider the size of a dihedral angle is to estimate the amount of rotation about the edge which is necessary in order to make one face coincide with the other; the greater the amount of rotation, the larger the angle.

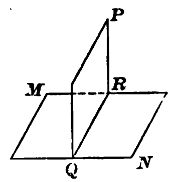
Two dihedral angles are *equal* when they can be placed so that their faces coincide.

Two dihedral angles are said to be *adjacent* when they have the same edge and a common face between them.

Two dihedral angles are said to be *vertical* when they have the same edge and the faces of one are the prolongations of the faces of the other.

The terms *acute, obtuse, oblique, complementary, supplementary, alternate interior, corresponding,* etc. can be applied to dihedral angles. The definitions are similar to those for the corresponding cases of angles in Plane Geometry.

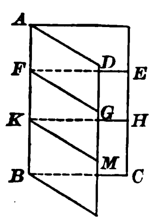
When one plane meets another plane so as to form two equal adjacent dihedral angles, each angle is called a *right dihedral angle,* and the planes are said to be *perpendicular* to each other. For example, the dihedral angles $PQRM$ and $PQRN$ are equal, and each is a right dihedral angle; the planes MN and PQ are perpendicular to each other.



The angle formed by two straight lines, one in each face of a dihedral angle, perpendicular to the edge at the same point, is called the *plane angle* of the dihedral angle.

Proposition 228 Theorem

All plane angles of a dihedral angle are equal.



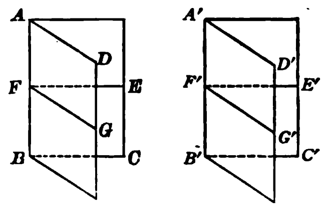
Hypothesis. $\angle s$ EFG and HKM are any two plane $\angle s$ of the dihedral $\angle AB$.

Conclusion. $\angle EFG = \angle HKM$.

HINT. Consult Prop. 32 and Prop. 222.

Proposition 229 Theorem

Two dihedral angles are equal when their plane angles are equal.



Hypothesis. The equal ∠s EFG and $E'F'G'$ are respectively plane angles of the dihedral ∠s AB and $A'B'$.

Conclusion. Dihedral $\angle\ AB$ = dihedral $\angle\ A'B'$.

Proof. Apply the dihedral $\angle\ AB$ to the dihedral $\angle\ A'B'$ so that $\angle\ EFG$ shall coincide with $\angle\ E'F'G'$, FE falling along $F'E'$, and FG along $F'G'$.

The plane determined by FE and FG coincides with the plane determined by $F'E'$ and $F'G'$. (?)

AB is $\perp$ to the plane determined by FE and FG, and $A'B'$ is $\perp$ to the plane determined by $F'E'$ and $F'G'$. (?)

Then AB coincides with $A'B'$. (?)

∴ planes AC and BD coincide with $A'C'$ and $B'D'$ respectively. (?)

∴ dihedral $\angle\ AB$ = dihedral $\angle\ A'B'$. (?)

Q. E. D.

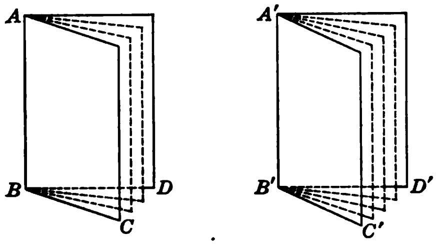
Cor. I. *Plane angles of equal dihedral angles are equal.*

Cor. II. *If two planes intersect each other, the vertical dihedral angles are equal.*

Ex. 994. The intersections of the faces of a dihedral angle with any plane perpendicular to the edge form a plane angle of the dihedral angle.

Proposition 230 Theorem

Two dihedral angles are in the same ratio as their plane angles.



CASE I. *When the plane angles are commensurable.*

Hypothesis. The commensurable ∠s CBD and $C'B'D'$ are respectively the plane ∠s of the dihedral ∠s $CABD$ and $C'A'B'D'$.

Conclusion. $$\frac{\text{Dihedral } \angle CABD}{\text{Dihedral } \angle C'A'B'D'} = \frac{\angle CBD}{\angle C'B'D'}.$$

Proof. Suppose a common measure of $\angle CBD$ and $\angle C'B'D'$ to be contained in $\angle CBD$ m times and in $\angle C'B'D'$ n times.

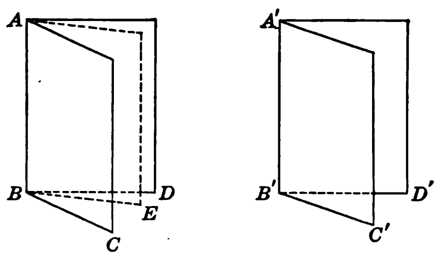
$$\text{Then } \frac{\angle CBD}{\angle C'B'D'} = \frac{m}{n}.$$

Through the several lines of division and the edges pass planes. These planes divide dihedral $\angle CABD$ into m parts and dihedral $\angle C'A'B'D'$ into n parts, all of which are equal. (?)

$$\therefore \frac{\text{dihedral } \angle CABD}{\text{dihedral } \angle C'A'B'D'} = \frac{m}{n}.$$

$$\therefore \frac{\text{dihedral } \angle CABD}{\text{dihedral } \angle C'A'B'D'} = \frac{\angle CBD}{\angle C'B'D'}. \quad (?)$$

CASE II. *When the plane angles are incommensurable.*



Hypothesis. The incommensurable ∠s CBD and $C'B'D'$ are respectively the plane ∠s of the dihedral ∠s $CABD$ and $C'A'B'D'$.

Conclusion. $$\frac{\text{Dihedral } \angle CABD}{\text{Dihedral } \angle C'A'B'D'} = \frac{\angle CBD}{\angle C'B'D'}.$$

Proof. Divide $\angle C'B'D'$ into any number of equal parts, and taking one of these parts as a unit of measure, apply it to $\angle CBD$. Since ∠s CBD and $C'B'D'$ are incommensurable, the unit taken a certain number of times will form the $\angle CBE$, leaving a remainder $\angle EBD$ less than the unit of measure.

Pass a plane through BE and AB.

Then $$\frac{\text{dihedral } \angle CABE}{\text{dihedral } \angle C'A'B'D'} = \frac{\angle CBE}{\angle C'B'D'}. \qquad \text{(CASE I.)}$$

If, now, the unit be subdivided and one of these subdivisions be applied to $\angle CBD$, the remainder will be much less than $\angle EBD$. By continuing this process of subdivision the remainder becomes still smaller, and the difference between $\frac{\angle CBE}{\angle C'B'D'}$ and $\frac{\angle CBD}{\angle C'B'D'}$, can be made so small as to become and remain less than any assigned value, however small.

At the same time the difference between $\frac{\text{dihedral } \angle CABE}{\text{dihedral } \angle C'A'B'D'}$ and $\frac{\text{dihedral } \angle CABD}{\text{dihedral } \angle C'A'B'D'}$ is becoming less and less, and this difference can be made so small as to become and remain less than any assigned value, however small.

Hence $\frac{\text{dihedral } \angle CABE}{\text{dihedral } \angle C'A'B'D'} \doteq \frac{\text{dihedral } \angle CABD}{\text{dihedral } \angle C'A'B'D'}$ and $\frac{\angle CBE}{\angle C'B'D'} \doteq \frac{\angle CBD}{\angle C'B'D'}$.

Now the variables $\frac{\text{dihedral } \angle CABE}{\text{dihedral } \angle C'A'B'D'}$ and $\frac{\angle CBE}{\angle C'B'D'}$ are always one and the same number.

Hence their limits are one and the same number, and

$$\frac{\text{dihedral } \angle CABD}{\text{dihedral } \angle C'A'B'D'} = \frac{\angle CBD}{\angle C'B'D'}.$$

Q. E. D.

DISCUSSION. *A dihedral angle is measured by its plane angle,* just as a central angle in a circle is measured by its intercepted arc (Prop. 126.) For example, if the plane angle of a dihedral angle is an angle of 30°, the dihedral angle is said to be an angle of 30°.

Many properties of dihedral angles analogous to those of plane angles can be proved by the use of this principle.

Ex. 995. Through any straight line in a plane one plane perpendicular to the given plane can be drawn, and only one.

Ex. 996. The sum of the two adjacent dihedral angles formed by one plane meeting another is equal to two right dihedral angles.

Ex. 997. If the sum of two adjacent dihedral angles is equal to two right dihedral angles, their exterior faces are in the same plane.

Ex. 998. If two parallel planes are cut by a third plane, the alternate interior dihedral angles are equal.

Ex. 999. If two parallel planes are cut by a third plane, the corresponding dihedral angles are equal.

Ex. 1000. If two planes are cut by a third plane so as to make the alternate interior dihedral angles equal, the two planes are parallel.

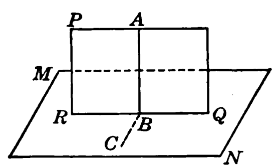
Ex. 1001. If two planes are cut by a third plane so as to make the corresponding dihedral angles equal; the two planes are parallel.

Ex. 1002. Two dihedral angles whose faces are respectively parallel to each other and whose edges are parallel are either equal or supplementary.

Ex. 1003. Two dihedral angles whose faces are respectively perpendicular to each other and whose edges are parallel are either equal or supplementary.

Proposition 231 Theorem

If a line is perpendicular to a plane, every plane passed through the line is perpendicular to the plane.



Hypothesis. The line AB is $\perp$ to the plane MN, and PQ is a plane passed through AB intersecting MN in the line RQ.

Conclusion. PQ is $\perp$ to MN.

Proof. Draw BC in the plane MN $\perp$ to RQ at B.

AB is $\perp$ to RQ. (?)

Then $\angle ABC$ is a plane $\angle$ of the dihedral $\angle PQRN$. (?)

But $\angle ABC$ is a rt. $\angle$. (?)

$\therefore$ dihedral $\angle PQRN$ is a rt. $\angle$, and PQ is $\perp$ to MN. (?)

Q. E. D.

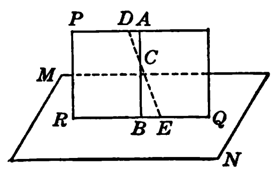
Proposition 232 Theorem

If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.

HINT. Use the same figure as in Prop. 231.

Proposition 233 Theorem

If two planes are perpendicular to each other, a line perpendicular to one of them through any point of the other lies in the other.



Hypothesis. The plane PQ is $\perp$ to the plane MN, and through C, a point in PQ, AB is drawn $\perp$ to MN.

Conclusion. AB lies in PQ.

HINT. Draw DE through C in the plane PQ $\perp$ to RQ, the intersection of MN and PQ, and prove that AB coincides with DE.

Ex. 1004. If a line is perpendicular to a plane, every plane parallel to the line is perpendicular to the plane.

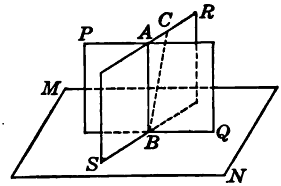
Ex. 1005. If a line and a plane are parallel, any plane perpendicular to the line is also perpendicular to the plane.

Ex. 1006. If a plane and a line not lying in it are both perpendicular to the same plane, they are parallel.

Ex. 1007. If a plane is perpendicular to one of two parallel planes, it is perpendicular to the other also.

Proposition 234 Theorem

If two intersecting planes are perpendicular to a third plane, their intersection is also perpendicular to that plane.



Hypothesis. The planes PQ and RS, which intersect in the line AB, are both $\perp$ to the plane MN.

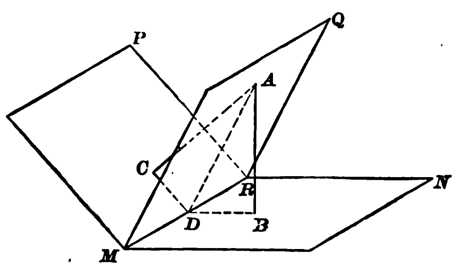
Conclusion. AB is $\perp$ to MN.

HINT. Draw BC $\perp$ to MN at B, the point common to the three planes, and prove that BC lies in both the planes PQ and RS.

COR. *If a plane is perpendicular to each of two intersecting planes, it is perpendicular to their intersection.*

Proposition 235 Theorem

Every point in the plane which bisects a dihedral angle is equally distant from the faces of the angle.



Hypothesis. *A* is any point in *MQ*, the plane bisecting the dihedral $\angle NMRP$, and *AB* and *AC* are the ⊥s drawn from *A* to the planes *MN* and *MP* respectively.

Conclusion. $AB = AC$.

Proof. Let the plane determined by *AB* and *AC* intersect the planes *MN*, *MQ*, and *MP* in *DB*, *DA*, and *DC* respectively.

The plane *ABC* is ⊥ to the planes *MN* and *MP*. (?)

∴ the plane *ABC* is ⊥ to *MR*. (?)

∴ *DB*, *DA*, and *DC* are ⊥ to *MR*. (?)

∴ the ∠s *ADB* and *ADC* are respectively the plane ∠s of the dihedral ∠s *QMRN* and *QMRP*. (?)

$\therefore \angle ADB = \angle ADC.$ (?)

$\triangle ADB \equiv \triangle ADC.$ (?)

$\therefore AB = AC.$ (?)

Q. E. D.

COR. I. *The locus of a point within a dihedral angle and equidistant from its faces is the plane which bisects the dihedral angle.*

COR. II. *The locus of a point equidistant from two intersecting planes is a pair of planes which bisect the dihedral angles formed by the given planes.*

Ex. 1008. If each of two intersecting planes is perpendicular to a third plane, every plane passing through the line in which the two planes intersect is also perpendicular to the third plane.

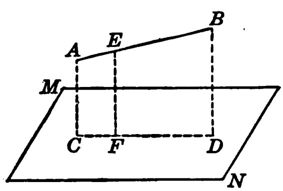
Ex. 1009. If two planes are respectively perpendicular to two intersecting lines, their line of intersection is perpendicular to the plane determined by the lines.

Ex. 1010. If from any point within a dihedral angle perpendiculars are drawn to the faces, the angle between these perpendiculars is the supplement of the plane angle of the dihedral angle.

Ex. 1011. From A, a point in one of two intersecting planes, AB is drawn perpendicular to the first plane, and AC perpendicular to the second; if these perpendiculars meet the second plane at B and C respectively, prove that BC is perpendicular to the intersection of the two planes.

Proposition 236 Theorem

Through a given line not perpendicular to a given plane one plane can be drawn perpendicular to the given plane, and only one.



Hypothesis. AB is a line not $\perp$ to the plane MN.

Conclusion. Through AB one plane can be drawn $\perp$ to MN, and only one.

Proof. From E, any point in AB, draw $EF \perp$ to MN, and let AD be the plane determined by AB and EF.

AD is $\perp$ to MN. (?)

If possible, suppose that a second plane can be drawn through $AB \perp$ to MN. This plane will intersect the plane AD in the line AB, and AB will be $\perp$ to MN. (?)

But this is contrary to the hypothesis that AB is not $\perp$ to MN.

$\therefore$ AD is the only plane that can be drawn through AB $\perp$ to MN.

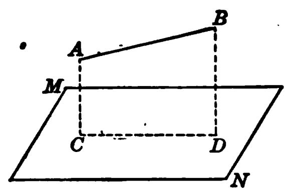
Q. E. D.

DEFINITIONS. The *projection of a point upon a plane* is the foot of the perpendicular drawn from the point to the plane.

The *projection of a line upon a plane* is the locus of the projections of all the points of the line upon the plane.

Proposition 237 Theorem

If a straight line is not perpendicular to a plane, its projection upon the plane is a straight line.



Hypothesis. AB is a straight line not ⊥ to the plane MN.

Conclusion. The projection of AB upon MN is a straight line.

Proof. The projection of AB upon MN is made up of the feet of the ⊥s drawn to MN from all the points in AB.

Through AB pass the plane AD ⊥ to MN.

Every ⊥ to MN drawn from a point in AB lies in the plane AD. (?)

∴ the projection of every point of AB upon MN lies in both the planes MN and AD, and the projection of AB upon MN is the intersection of these planes.

Now the intersection of two planes is a straight line. (?)

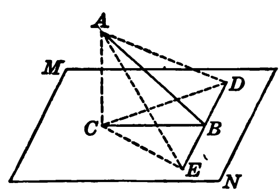
∴ the projection of AB upon MN is a straight line.

Q. E. D.

Definition. The plane determined by a line and the perpendicular to a plane drawn from any point in the line is called the *projecting plane* of the line. In the figure of Prop. 237, the plane AD is the projecting plane of the line AB upon the plane MN. This plane contains all the perpendiculars drawn to MN from points in AB.

Proposition 238 Theorem

If the projection of a line upon a plane is perpendicular to a line in the plane, the line itself is perpendicular to the line in the plane.



Hypothesis. CB, the projection of AB upon the plane MN, is $\perp$ to DE, a line in the plane.

Conclusion. AB is $\perp$ to DE.

Proof. Take $BD = BE$, and draw AD, AE, CD, and CE.

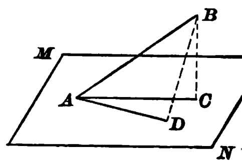
[To be completed by the student.]

Ex. 1012. State and prove the converse of Prop. 238.

Ex. 1013. Two planes intersect in the line CD. If the projection of a line AB upon one of these planes is perpendicular to CD, prove that the projection of AB upon the other plane is also perpendicular to CD.

Proposition 239 Theorem

The acute angle which a line makes with its projection upon a plane is the least angle which it makes with any line in the plane.



Hypothesis. AC is the projection of the line AB upon the plane MN, and AD is any other line drawn through A in the plane.

Conclusion. $\angle BAC < \angle BAD$.

Proof. Take $AD = AC$, and draw BC and BD.

[To be completed by the student.]

DEFINITION. The *angle which a line makes with a plane* is the angle formed by the line and its projection upon the plane; this angle is called the *inclination of the line to the plane*.

Ex. 1014. If two lines are parallel, their projections on any plane are either the same line or parallel lines.

Ex. 1015. If two parallel lines intersect a plane, they make equal angles with it.

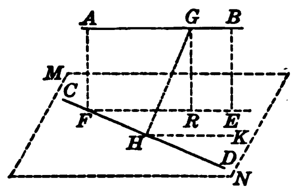
Ex. 1016. If a line intersects two parallel planes, it makes equal angles with them.

Ex. 1017. Equal oblique lines drawn to a plane from a point without it are equally inclined to the plane.

Ex. 1018. AC is the projection of the line AB upon the plane MN, and AD and AE are two lines drawn in MN making equal angles with AC. Prove that AD and AE make equal angles with AB.

Proposition 240 Theorem

Between two skew lines one common perpendicular can be drawn, and only one.



Hypothesis. AB and CD are two skew lines.

Conclusion. One common $\perp$ to AB and CD can be drawn, and only one.

Proof. Through CD pass the plane $MN \parallel$ to AB.

Through AB pass the plane $AE \perp$ to MN, intersecting MN in the line FE.

FE is $\parallel$ to AB. (?)

FE is not $\parallel$ to CD. (?)

Let F be the intersection of CD and FE.

At F erect $FA \perp$ to FE in the plane AE.

FA is $\perp$ to MN. (?)

$\therefore FA$ is $\perp$ to CD. (?)

FA is $\perp$ to AB. (?)

Hence FA is a common $\perp$ to AB and CD.

If possible, suppose that GH is another common $\perp$ to AB and CD.

Let the plane determined by AB and GH intersect MN in the line HK.

AB is $\parallel$ to HK. (?)

$\therefore GH$ is $\perp$ to HK. (?)

$\therefore GH$ is $\perp$ to MN. (?)

But GR, drawn in the plane $AE \perp$ to FE, is $\perp$ to MN. (?)

Then there are two ⊥s from G to MN.

But this is impossible. (?)

$\therefore GH$ is not a common $\perp$ to AB and CD, and FA is the only common $\perp$.

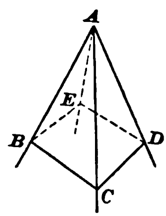
Q.E.D.

COR. *The common perpendicular is the shortest line that can be drawn between two skew lines.* $GR < GH$; $\therefore AF < GH$.

POLYHEDRAL ANGLES

When three or more planes meet at a common point, they are said to form a *polyhedral angle*, or *solid angle*. The common point is called the *vertex* of the angle, the intersections of the planes are called the *edges* of the angles, the portions of the planes included between the edges are called the *faces*, and the angles formed by the edges are called the *face angles*.

For example, the planes BAC, CAD, DAE, and EAB, meeting at the common point A, form the polyhedral angle A-$BCDE$. A is the vertex; AB, AC, AD, and AE are the edges; the planes BAC, CAD, DAE, and EAB are the faces; and the angles BAC, CAD, DAE, and EAB are the face angles.



A polyhedral angle of three faces is called a *trihedral* angle; a polyhedral angle of four faces is called a *tetrahedral* angle, and so on.

A trihedral angle is called *rectangular*, *bi-rectangular*, or *tri-rectangular*, according as it has one, two, or three right dihedral angles.

A trihedral angle is called *isosceles* when two of its face angles are equal.

Two polyhedral angles are called *vertical* when they have a common vertex and the edges of one are the prolongations of the edges of the other.

The polygon formed by the intersections of the faces of a polyhedral angle with a plane cutting all the edges is called a *section* of the polyhedral angle. In the figure on page 307, *BCDE* is a section of the polyhedral angle *A-BCDE.*

A polyhedral angle is said to be *convex* when any section is a convex polygon.

The face angles and the dihedral angles formed by the faces are called the *parts* of a polyhedral angle.

Two polyhedral angles are congruent when the parts of one are equal respectively to the parts of the other and are arranged in the same order, for one may be applied to the other so that they will coincide.

Two polyhedral angles are said to be *symmetrical* when the parts of one are equal respectively to the parts of the other, but are arranged in reverse order. In general, two symmetrical polyhedral angles do not coincide when one is applied to the other.

Many properties of trihedral angles are analogous to properties of triangles. A number of theorems of Plane Geometry concerning triangles may be changed to theorems concerning trihedral angles by changing *angle* and *side* to *dihedral angle* and *face angle* respectively.

Proposition 241 Theorem

Two vertical polyhedral angles are symmetrical.

Consult Prop. 5 and Prop. 229, Cor. II.

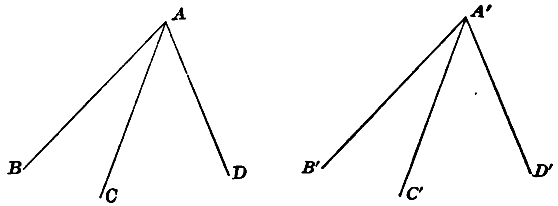
Ex. 1019. In an isosceles trihedral angle, the dihedral angles opposite the equal face angles are equal.

Ex. 1020. State and prove the converse of Ex. 1019.

Proposition 242 Theorem

Two trihedral angles are either congruent or symmetrical when two face angles and the included dihedral angle of one are equal respectively to two face angles and the included dihedral angle of the other.

CASE I. *When the equal parts are arranged in the same order, the trihedral angles are congruent.*



Hypothesis. In the trihedral ∡ $A\text{-}BCD$ and $A'\text{-}B'C'D'$, $\angle BAD = \angle B'A'D'$, $\angle BAC = \angle B'A'C'$, and dihedral $\angle AB =$ dihedral $\angle A'B'$.

Conclusion. Trihedral $\angle A\text{-}BCD \equiv$ trihedral $\angle A'\text{-}B'C'D'$.

Proof. Apply $A\text{-}BCD$ to $A'\text{-}B'C'D'$ so that $\angle BAD$ shall coincide with $\angle B'A'D'$, AB falling along $A'B'$ and AD along $A'D'$.

Face BAC will fall on face $B'A'C'$. (?)

Edge AC will fall along edge $A'C'$. (?)

Face CAD will fall on face $C'A'D'$. (?)

∴ trihedral $\angle A\text{-}BCD \equiv$ trihedral $\angle A'\text{-}B'C'D'$. (?)

CASE II. *When the equal parts are arranged in reverse order, the trihedral angles are symmetrical.*

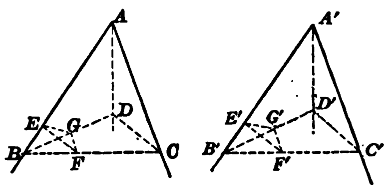
HINT. Produce the edges of one of the given trihedral angles through the vertex, thus forming a trihedral angle which can be proved congruent to the other given trihedral angle.

Proposition 243 Theorem

Two trihedral angles are either congruent or symmetrical when a face angle and the two adjacent dihedral angles of one are equal respectively to a face angle and the two adjacent dihedral angles of the other.

Proposition 244 Theorem

If two trihedral angles have the three face angles of one equal respectively to the three face angles of the other, the homologous dihedral angles are equal.



Hypothesis. In the trihedral $\measuredangle$ $A\text{-}BCD$ and $A'\text{-}B'C'D'$, $\angle BAC = \angle B'A'C'$, $\angle CAD = \angle C'A'D'$, and $\angle BAD = \angle B'A'D'$.

Conclusion. Dihedral $\angle AB$ = dihedral $\angle A'B'$, dihedral $\angle AC$ = dihedral $\angle A'C'$, and dihedral $\angle BC$ = dihedral $\angle B'C'$.

Proof. On the edges take $AB = AC = AD = A'B' = A'C' = A'D'$.

Draw BC, CD, BD, $B'C'$, $C'D'$, and $B'D'$.

$$\triangle BAC \cong \triangle B'A'C' \qquad (?)$$

$$\therefore BC = B'C' \qquad (?)$$

In like manner it can be proved that $BD = B'D'$ and $CD = C'D'$.

$$\therefore \triangle BCD \cong \triangle B'C'D'. \qquad (?)$$

At E, any point in AB, erect $EF \perp$ to AB in the plane BAC.

$$\angle ABC \text{ is an acute } \angle. \qquad (?)$$

$$\therefore EF \text{ meets } BC. \qquad (?)$$

Let F be the point of meeting.

Likewise, erect $EG \perp$ to AB in the plane BAD, meeting BD at G.

Draw FG.

On $A'B'$ take $B'E' = BE$, and construct $\triangle E'F'G'$ in the same manner that $\triangle EFG$ was constructed.

$$\triangle BEF \equiv \triangle B'E'F'. \qquad (?)$$

$$\therefore BF = B'F', \text{ and } EF = E'F'. \qquad (?)$$

In like manner it be can proved that $BG = B'G'$, and $EG = E'G'$.

$$\triangle BFG \equiv \triangle B'F'G'. \qquad (?)$$

$$\therefore FG = F'G'. \qquad (?)$$

$$\triangle EFG \equiv \triangle E'F'G'. \qquad (?)$$

$$\therefore \angle FEG = \angle F'E'G'. \qquad (?)$$

$\angle$s FEG and $F'E'G'$ are respectively the plane $\angle$s of the dihedral $\angle$s AB and $A'B'$. (?)

$$\therefore \text{dihedral } \angle AB = \text{dihedral } \angle A'B'. \qquad (?)$$

In like manner it can be proved that dihedral $\angle AC$ = dihedral $\angle A'C'$, and dihedral $\angle BC$ = dihedral $\angle B'C'$.

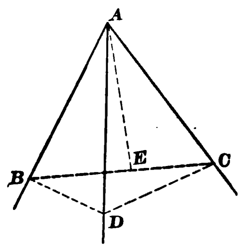
Q. E. D.

COR. *Two trihedral angles are either congruent or symmetrical when the three face angles of one are equal respectively to the three face angles of the other.* The trihedral angles are congruent if the parts are arranged in the same order; they are symmetrical if the parts are arranged in reverse order.

Ex. 1021. If two isosceles trihedral angles have the three face angles of one equal respectively to the three face angles of the other, they are congruent.

Proposition 245 Theorem

The sum of any two face angles of a trihedral angle is greater than the third face angle.



Hypothesis. In the trihedral $\angle$ *A-BCD*, $\angle BAC$ is the greatest face $\angle$.

Conclusion. $\angle BAD + \angle DAC > \angle BAC$.

Proof. In the face *BAC* draw *AE*, making $\angle BAE = \angle BAD$.

Take $AD = AE$, and through *D* and *E* pass a plane intersecting *AB* at *B* and *AC* at *C*, thus forming the section *BDC*.

$$\triangle BAD \equiv \triangle BAE. \quad (?)$$

$$\therefore BD = BE. \quad (?)$$

$$BD + DC > BE + EC. \quad (?)$$

$$\therefore DC > EC. \quad (?)$$

$$\therefore \angle DAC > \angle EAC. \quad (?)$$

$$\therefore \angle BAD + \angle DAC > \angle BAC. \quad (?)$$

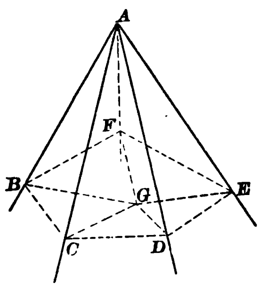
Q. E. D.

Ex. 1022. The three planes bisecting the dihedral angles of a trihedral angle intersect in a common straight line.

Ex. 1023. Find the locus of a point equidistant from the faces of a trihedral angle.

Proposition 246 Theorem

The sum of the face angles of any convex polyhedral angle is less than four right angles.



Hypothesis. $A\text{-}BCDEF$ is a convex polyhedral $\angle$.

Conclusion. $\angle BAC + \angle CAD +$ etc. < 4 rt. $\angle$s.

Proof. Let the polygon $BCDEF$ be a section of the polyhedral $\angle A\text{-}BCDEF$.

From G, any point within the polygon, draw GB, GC, GD, etc.

Then the number of $\triangle$s having the common vertex G is the same as the number of $\triangle$s having the common vertex A.

The sum of all the $\angle$s of the $\triangle$s having the common vertex A is equal to the sum of all the $\angle$s of the $\triangle$s having the common vertex G. (?)

Now $\angle ABF + \angle ABC > \angle GBF + GBC$,

$\angle ACB + \angle ACD > \angle GCB + \angle GCD$, etc. (?)

$\therefore$ the sum of the base $\angle$s of the $\triangle$s having the common vertex A is greater than the sum of the base $\angle$s of the $\triangle$s having the common vertex G. (?)

$\therefore$ the sum of the $\angle$s at $A <$ the sum of the $\angle$s at G. (?)

But the sum of the $\angle$s at $G = 4$ rt. $\angle$s. (?)

$\therefore$ the sum of the $\angle$s at $A < 4$ rt. $\angle$s. (?)

Q. E. D.

EXERCISES

Ex. 1024. How many horizontal planes can be drawn through a horizontal line? How many vertical planes can be drawn through a vertical line?

Ex. 1025. If three planes mutually intersect, the three lines of intersection are either parallel or concurrent.

Ex. 1026. If a line is parallel to a plane, it is everywhere equally distant from the plane.

Ex. 1027. If a line is parallel to each of two planes, the intersections which any plane passing through this line makes with the planes are parallel.

Ex. 1028. If two lines are parallel, the line of intersection of two planes drawn through them is parallel to each of the lines.

Ex. 1029. Show that all the lines which intersect one given line parallel to a second given line lie in one plane parallel to the second line.

Ex. 1030. If a plane is parallel to one of two mutually perpendicular planes, it is perpendicular to the other.

Ex. 1031. Is Ex. 1030 true when the words perpendicular and parallel are interchanged throughout? Prove your answer.

Ex. 1032. If two planes are determined by two parallel lines and a point without the plane of these lines, the intersection of the two planes is parallel to each of the given lines.

Ex. 1033. If two lines are respectively perpendicular to two intersecting planes, they make equal angles with the planes to which they are not perpendicular.

Ex. 1034. If the intersections of several planes are parallel, all the perpendiculars to these planes drawn from the same point in space lie in one plane.

Ex. 1035. If two intersecting planes are cut by two parallel planes not parallel to their line of intersection, the lines of intersection form equal angles.

Ex. 1036. In any skew quadrilateral (that is, a quadrilateral whose sides do not lie in the same plane), if lines are drawn joining the midpoints of the sides, the figure thus formed is a parallelogram.

Ex. 1037. The sum of the angles of a skew quadrilateral is less than four right angles.

Ex. 1038. If from any point within a dihedral angle perpendiculars are drawn to the faces, the plane determined by these perpendiculars is perpendicular to the edge of the dihedral angle.

Ex. 1039. MN and PQ are parallel planes. A is a point in MN and B is a point in PQ such that the line AB is perpendicular to MN. C is another point in MN and D is another point in PQ such that CD is perpendicular to PQ. Prove that the lines AB and CD are parallel.

Ex. 1040. The planes MN and PQ intersect in the line AB; from C, any point in MN, CD and CE are drawn perpendicular to AB and PQ respectively, meeting AB and PQ at D and E respectively. Prove that AB is perpendicular to the plane determined by CD and CE.

Ex. 1041. AB, AC, and AD are perpendicular to each other at the common point A. If BC is drawn, and AE is drawn perpendicular to BC, prove that the line joining D and E is perpendicular to BC.

Ex. 1042. The line DE is perpendicular to the plane of the triangle ABC at the centre of the circumscribed circle. Prove that a line drawn from any point in DE to A is perpendicular to the tangent to the circle drawn through A.

Ex. 1043. A is a point without the plane MN, and AB and AC are two lines parallel to MN. If planes are passed through A perpendicular to AB and AC respectively, prove that their intersection is perpendicular to MN.

Ex. 1044. If the projections of a number of points lie in a straight line, these points lie in one plane.

Ex. 1045. The plane determined by a line and its projection upon a plane is perpendicular to the given plane.

Ex. 1046. If the projections of any line upon two planes which are not parallel are both straight lines, the given line is a straight line.

Ex. 1047. If a line is perpendicular to one of two intersecting planes, its projection on the other plane is perpendicular to the line of intersection of the two planes.

Ex. 1048. A plane parallel to each of two lines and bisecting the common perpendicular to these lines, bisects every line joining a point of one of these lines to a point of the other.

Ex. 1049. A, B, C, and D are four points in a plane such that $AB = AC = AD$, and E is a point without the plane such that $EB = EC = ED$; prove that EA is perpendicular to the plane.

Ex. 1050. The perpendiculars common to a line x and to each one of a set of parallel lines skew to x are constructed. Prove that these perpendiculars all lie in one plane. Can two of the parallel lines have their common perpendiculars to x along the same line?

Ex. 1051. An isosceles trihedral angle and its symmetrical trihedral angle are congruent.

Ex. 1052. If two face angles of a trihedral angle are unequal, the dihedral angles opposite them are unequal, and the greater dihedral angle is opposite the greater face angle.

Ex. 1053. State and prove the converse of Ex. 1052.

Ex. 1054. In any trihedral angle, the three planes passing through the edges and the bisectors of the opposite face angles intersect in a common straight line.

Ex. 1055. In any trihedral angle, the three planes passing through the bisectors of the face angles, and perpendicular to these faces respectively, intersect in a common straight line.

Ex. 1056. In any trihedral angle, the three planes passing through the edges perpendicular to the opposite sides intersect in a common straight line.

Ex. 1057. In the trihedral angle A-BCD, the line AE bisects the face angle BAC; prove that the angle DAE is less than half the sum of the angles BAD and CAD.

Ex. 1058. Through a given point in a plane, to draw a line in that plane which shall be at a given distance from a given point without the plane.

Ex. 1059. Through a given point A to draw to a given plane a line which shall be parallel to a given plane MN and of given length.

Ex. 1060. To cut a polyhedral angle of four faces by a plane so that the section shall be a parallelogram.

Ex. 1061. Find the locus of a point which is equidistant from two given points, and at the same time is equidistant from two given parallel planes.

Ex. 1062. Find the locus of a point which is equidistant from two given points, and at the same time is equidistant from two given intersecting planes.

Ex. 1063. Find the locus of a point which is at a given distance from a given plane, and at the same time is equidistant from two given intersecting planes.

Ex. 1064. Find the locus of lines through the point A which make equal angles with the lines AB and AC.

Ex. 1065. AB and CD are two skew lines; E is any point in AB and F is any point in CD. Find the locus of the mid-point of the line EF.

Ex. 1066. Find the locus of any given point on a line of fixed length, the ends of which lie in parallel planes.

Ex. 1067. Find the locus of a point in space equidistant from two intersecting lines.

Ex. 1068. Find the locus of a point which moves so that the ratio of its distance from two given points is constant.

Ex. 1069. Find the locus of a point equidistant from the edges of a trihedral angle.

PROBLEMS OF COMPUTATION

1. B is the projection of the point A on the plane MN, and C is another point in the same plane. If the distances from C to A and B are 15 in. and 12 in. respectively, find the distance from A to the plane.

2. A line 6 ft. long meets a plane at an angle of 45°; find the length of its projection on the plane.

3. If a line meets a plane at an angle of 60°, and the length of its projection on the plane is 10 in., what is the length of the line?

4. A pole 10 ft. long reaches from the ceiling to the floor of a room and meets the floor at a point 6 ft. from the foot of the perpendicular drawn from the other end of the pole to the floor. Find the height of the room.

5. The distances of two points, A and B, from a given plane are 11 in. and 19 in. respectively, and the distance between the feet of the perpendiculars drawn from A and B to the plane is 6 in. Find the distance from A to B.

6. The line AC meets three parallel planes in the points A, B, and C; the line DF meets the same planes in the points D, E, and F. If $AC = 12$ in., $AB = 8$ in., and $EF = 6$ in., find the value of DF.

Book VII

POLYHEDRONS

A *polyhedron* is a solid bounded by planes. The intersections of the planes are called the *edges* of the polyhedron, the intersections of the edges are called the *vertices*, and the portions of the planes bounded by the edges are called the *faces*.

A straight line joining any two vertices not in the same face is called a *diagonal* of the polyhedron.

A polyhedron must have at least four faces. The least number of planes that can form a polyhedral angle is three, and it requires one plane in addition to these to enclose a definite portion of space.

A polyhedron of four faces is called a *tetrahedron;* one of six faces, a *hexahedron;* one of eight faces, an *octahedron;* one of twelve faces, a *dodecahedron;* one of twenty faces, an *icosahedron.*

The polygon formed by the intersection of a plane with three or more faces of a polyhedron is called a *section* of the polyhedron.

A polyhedron is said to be *convex* when every section is a convex polygon.

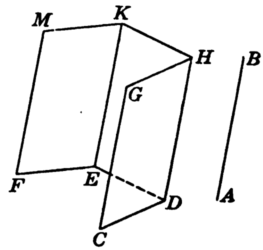
NOTE. Whenever the word *polyhedron* is used alone in this book, a convex polyhedron is meant.

The *volume* of a solid is the ratio of the solid to another solid, called the *unit of volume.* For example, if V is a certain volume and U is the unit of volume, the volume of V is $\frac{V}{U}$.

Solids that have equal volumes are said to be *equivalent.*

PRISMS AND PARALLELEPIPEDS

A *prismatic surface* is a surface generated by a straight line which continually moves along a fixed broken line and is constantly parallel to a given straight line not coplanar with the broken line.



For example, if the straight line CG moves along the broken line $CDEF$ so that in every position it is parallel to AB, the surface thus generated is a prismatic surface.

The moving line is called the *generatrix* and the broken line is called the *directrix*.

It follows from the definition that a prismatic surface is composed of planes, the intersections of which are parallel lines.

A *prism* is a polyhedron bounded by a closed prismatic surface and two parallel planes. The sections of the prismatic surface formed by the parallel planes are called the *bases* of the prism, the faces formed on the prismatic surface are called the *lateral faces*, the intersections of the lateral faces are called the *lateral edges*, and the sum of the areas of the lateral faces is called the *lateral area*.

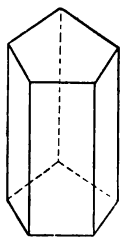
A prism is called *triangular*, *quadrangular*, *pentagonal*, etc., according as its bases are triangles, quadrilaterals, pentagons, etc.

A prism whose lateral edges are perpendicular to its bases is called a *right* prism. A prism whose lateral edges are oblique to its bases is called an *oblique* prism. A right prism whose bases are regular polygons is called a *regular* prism.

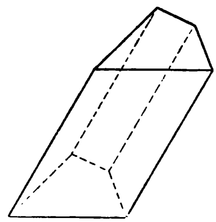
The perpendicular distance between the bases is called the *altitude* of the prism.

A section of a prism made by a plane perpendicular to the lateral edges is called a *right section.*

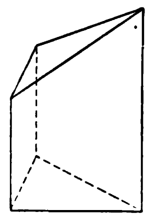
A *truncated prism* is the part of a prism included between either base and a section not parallel to the base, cutting all the lateral edges.



Right pentagonal prism



Oblique quadrangular prism



Truncated triangular prism

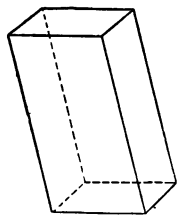
A prism whose bases are parallelograms is called a *parallelepiped.* Accordingly, a parallelepiped is a hexahedron having six parallelograms for its faces.

A parallelepiped whose lateral edges are perpendicular to its bases is called a *right* parallelepiped. A parallelepiped whose lateral edges are oblique to its bases is called an *oblique* parallelepiped. A right parallelepiped whose bases are rectangles is called a *rectangular* parallelepiped.

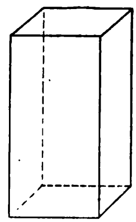
NOTE. A rectangular parallelepiped is sometimes called a *cuboid.*

A rectangular parallelepiped, all of whose edges are equal, is called a *cube.* The usual unit of volume is a cube, each edge of which is a linear unit.

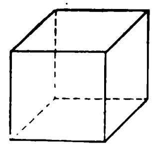
A parallelepiped has eight vertices, twelve edges, and four diagonals. The edges can be divided into three sets, each set being made up of four lines which are parallel and equal. Each diagonal is drawn from a vertex of one face to the opposite vertex of the opposite face.



Oblique parallelepiped



Right parallelepiped



Cube

The following principles follow at once from the preceding definitions:

(i) *The lateral edges of a prism are parallel and equal.*

(ii) *Any lateral edge of a right prism is equal to the altitude.*

(iii) *The lateral faces of a prism are parallelograms.*

(iv) *The lateral faces of a right prism are rectangles.*

(v) *All the faces of a parallelepiped are parallelograms.*

(vi) *All the faces of a rectangular parallelepiped are rectangles.*

(vii) *All the faces of a cube are congruent squares.*

(viii) *The opposite faces of a parallelepiped are congruent parallelograms, and their planes are parallel.*

(ix) *Any two opposite faces of a parallelepiped may be taken as the bases.*

Ex. 1070. The four diagonals of a parallelepiped are concurrent and mutually bisect each other.

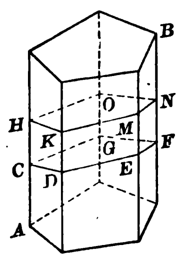
Ex. 1071. The four diagonals of a rectangular parallelepiped are equal.

Ex. 1072. The square of a diagonal of a rectangular parallelepiped is equal to the sum of the squares of any three concurrent edges.

Ex. 1073. The sum of the squares of the four diagonals of any parallelepiped is equal to the sum of the squares of its twelve edges.

Proposition 247 Theorem

Sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.



Hypothesis. $CDEFG$ and $HKMNO$ are sections of the prism AB made by two ∥ planes cutting all the lateral edges.

Conclusion. Polygon $CDEFG \equiv$ polygon $HKMNO$.

Proof. CD is ∥ to HK, DE is ∥ to KM, etc. (?)

$CD = HK$, $DE = KM$, etc. (?)

$\angle CDE = \angle HKM$, $\angle DEF = \angle KMN$, etc. (?)

$\therefore$ polygon $CDEFG \equiv$ polygon $HKMNO$. (?)

Q. E. D

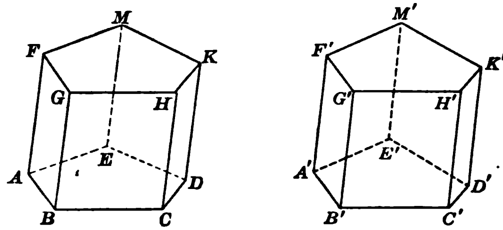
Cor. I. *The bases of a prism are congruent polygons.*

Cor. II. *Any section of a prism made by a plane parallel to the bases is congruent to the bases.*

Cor. III. *All right sections of a prism are congruent.*

Proposition 248 Theorem

Two prisms are congruent when the three faces including a trihedral angle of one are congruent respectively to the three faces including a trihedral angle of the other, and are similarly placed.



Hypothesis. In the prisms AK and $A'K'$, the faces AD, AG, and AM are congruent respectively to the faces $A'D'$, $A'C'$, and $A'M'$, and are similarly placed.

Conclusion. Prism $AK \equiv$ prism $A'K'$.

Proof. $\angle BAE = \angle B'A'E'$, $\angle BAF = \angle B'A'F'$, and $\angle EAF = \angle E'A'F'$. (?)

$\therefore$ trihedral $\angle A\text{-}BEF \equiv$ trihedral $\angle A'\text{-}B'E'F'$. (?)

Apply the prism AK to the prism $A'K'$ so that the trihedral $\angle A\text{-}BEF$ shall coincide with the trihedral $\angle A'\text{-}B'E'F'$, face AD falling on face $A'D'$, face AG on face $A'G'$, and face AM on face $A'M'$.

Polygon AD will coincide with polygon $A'D'$, $\square AG$ will coincide with $\square A'G'$, and $\square AM$ will coincide with $\square A'M'$. (?)

Plane FK will fall on plane $F'K'$. (?)

CH will fall along $C'H'$. (?)

$\therefore H$ will fall on H'. (?)

In like manner it may be proved that each of the vertices of FK will fall on the homologous vertex of $F'K'$.

$\therefore$ the prisms coincide and are congruent.

Q. E. D.

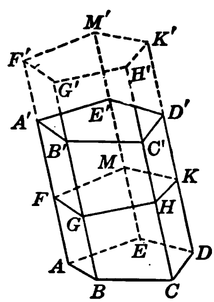
COR. I. *Two truncated prisms are congruent when the three faces including a trihedral angle of one are congruent respectively to the three faces including a trihedral angle of the other, and are similarly placed.* The preceding demonstration applies equally well to two truncated prisms.

COR. II. *Two right prisms having congruent bases and equal altitudes are congruent.* If the faces are not similarly placed, they will become so by inverting one of the prisms.

Ex. 1074. Two triangular prisms are congruent when the lateral faces of one are congruent respectively to the lateral faces of the other, and are similarly placed.

Proposition 249 Theorem

An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.



Hypothesis. FK is a right section of the oblique prism AD', and AA' is a lateral edge.

Conclusion. The oblique prism AD' is equivalent to a right prism whose base is FK and whose altitude is equal to AA'.

Proof. Produce AA' to F', making $FF' = AA'$. Let $F'K'$, the plane passing through $F' \perp$ to AF', intersect the lateral faces produced, forming the right section $F'K'$.

Then FK' is a right prism whose base is a right section of the oblique prism and whose altitude FF' is equal to a lateral edge of the oblique prism.

$$AA' = FF' \quad (?)$$

$$AF = A'F' \quad (?)$$

In like manner it can be proved that $BG = B'G'$, $CH = C'H'$, etc.

$$FK \equiv F'K' \quad (?)$$

Apply truncated prism AK to truncated prism $A'K'$ so that FK shall coincide with $F'K'$.

FA will fall along $F'A'$. (?)

A will fall on A'. (?)

In like manner it can be proved that B will fall on B', C on C', and so on.

$\therefore$ truncated prism $AK \equiv$ truncated prism $A'K'$. (?)

$\therefore$ oblique prism AD' = right prism FK'. (?)

Q. E. D.

Ex. 1075. Every lateral edge of a prism is parallel to the faces not passing through this edge.

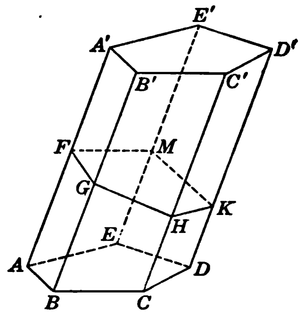
Ex. 1076. The lateral edges of a prism make equal angles with the bases.

Ex. 1077. Every section of a prism made by a plane parallel to a lateral edge is a parallelogram.

Ex. 1078. Every section of a prism made by a plane parallel to a lateral face is a parallelogram.

Proposition 250 Theorem

The lateral area of a prism is equal to the product of the perimeter of a right section and a lateral edge.



Hypothesis. Let FK be a right section of the prism AD'. Let S denote the lateral area of the prism, p the perimeter of a right section, and e a lateral edge.

Conclusion. $S = p \times e$.

Proof. $AA' = BB' = CC'$, etc. $= e$. (?)

FG is $\perp$ to AA', GH is $\perp$ to BB', etc. (?)

Area of $\square AB' = FG \times AA' = FG \times e$,

Area of $\square BC' = GH \times BB' = GH \times e$, etc. (?)

$\therefore S = (FG + GH +$ etc.$) \times e$. (?)

$\therefore S = p \times e$. (?)

Q. E. D.

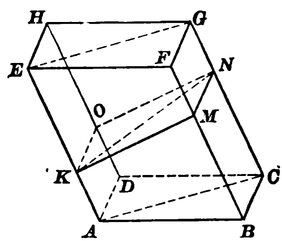
Cor. *The lateral area of a right prism is equal to the product of the perimeter of its base and its altitude.*

Ex. 1079. The area of any lateral face of a prism is less than the sum of the areas of the other lateral faces.

Ex. 1080. The lateral areas of right prisms of equal altitudes are in the same ratio as the perimeters of their bases.

Proposition 251 Theorem

The plane passed through two diagonally opposite edges of a parallelepiped divides it into two equivalent triangular prisms.



Hypothesis. The plane AG passes through the diagonally opposite edges AE and CG of the parallelepiped AG.

Conclusion. Prism $ABC\text{-}F$ = prism $ADC\text{-}H$.

Proof. Let $KMNO$ be a right section of the parallelepiped, cutting the plane AG in the line KN.

KM is ∥ to ON and MN is ∥ to KO. (?)

$\therefore KMNO$ is a ▱. (?)

$\therefore \triangle KMN \equiv \triangle KON$. (?)

Prism $ABC\text{-}F$ is equivalent to a right prism whose base is KMN and whose altitude is AE; and prism $ADC\text{-}H$ is equivalent to a right prism whose base is KON and whose altitude is AE.

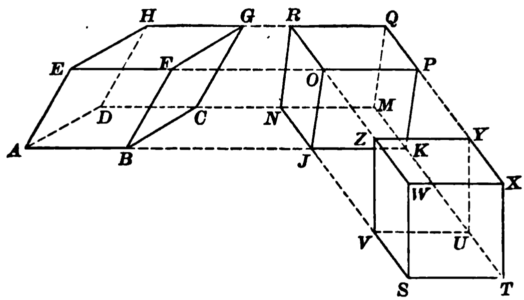
But these two right prisms are equal. (?)

$\therefore$ prism $ABC\text{-}F$ = prism $ADC\text{-}H$. (?)

Q. E. D

Proposition 252 Theorem

An oblique parallelepiped is equivalent to rectangular parallelepiped having an equivalent base and the same altitude.



Hypothesis. AG is an oblique parallelepiped whose base is $ABCD$ and whose altitude is h.

Conclusion. AG is equivalent to a rectangular parallelepiped whose base is equivalent to $ABCD$ and whose altitude is h.

Proof. Produce the edges AB, DC, EF, and HG, and on AB produced take $JK = AB$.

Let JQ be the right parallelepiped formed by right sections passing through J and K.

Then $JQ = AG$. (?)

Produce the edges NJ, MK, RO, and QP, and on NJ produced take $VS = NJ$.

Let SY be the right parallelepiped formed by right sections passing through V and S.

Then $SY = JQ$. (?)

$\therefore SY = AG$. (?)

Since the plane RS is $\perp$ to JK, the plane RS is $\perp$ to the plane NT. (?)

$\angle ZVU$ is the plane $\angle$ of the dihedral $\angle RSNT$. (?)

$\angle ZVU$ is a rt. $\angle$, and $VUYZ$ is a $\square$. (?)

$\therefore SY$ is a rectangular parallelepiped. (?)

$\square ABCD = \square JKMN = \square STUV$. (?)

The altitude of SY is h. (?)

Hence SY, which is equivalent to AG, has a base equivalent to $ABCD$ and its altitude is h.

Q. E. D.

Proposition 253 Theorem

Two rectangular parallelepipeds having congruent bases are to each other as their altitudes.

CASE I. *When the altitudes are commensurable.*

CASE II. *When the altitudes are incommensurable.*

Use the method of proof given in Prop. 230, and consult Prop. 248, Cor. II.

DEFINITION. The lengths of the three edges of a rectangular parallelepiped which meet at any vertex are called its *dimensions.*

REMARK. Theorem 253 may be stated as follows:

Two rectangular parallelepipeds having two dimensions in common are to each other as the third dimensions.

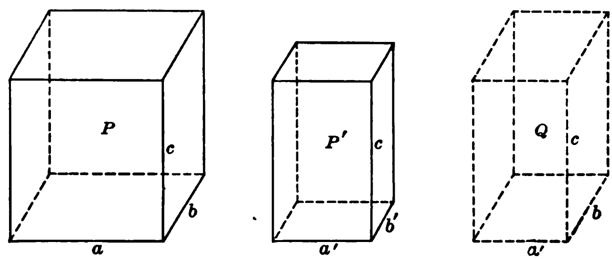
Ex. 1081. If three edges of a parallelepiped are concurrent and are mutually perpendicular to each other, the parallelepiped is rectangular.

Ex. 1082. If the two planes which are determined by the opposite lateral edges of a parallelepiped are perpendicular to the base, the parallelepiped is a right parallelepiped.

Ex. 1083. If the four diagonals of a quadrangular prism are concurrent, the prism is a parallelepiped.

Proposition 254 Theorem

Two rectangular parallelepipeds having equal altitudes are to each other as their bases.



Hypothesis. P and P' are two rectangular parallelepipeds whose respective dimensions are a, b, and c and a', b', and c, the altitude c being the same for both.

Conclusion. $$\frac{P}{P'} = \frac{a \times b}{a' \times b'}.$$

Proof. Construct the rectangular parallelepiped Q with dimensions equal to a', b, and c.

Then $\dfrac{P}{Q} = \dfrac{a}{a'}$,

and $\dfrac{Q}{P'} = \dfrac{b}{b'}$. (?)

$\therefore \dfrac{P}{P'} = \dfrac{a \times b}{a' \times b'}$. (?)

Q. E. D.

REMARK. Theorem 254 may be stated as follows:
Two rectangular parallelepipeds having one dimension in common are to each other as the products of the other two dimensions.

Proposition 255 Theorem

Two rectangular parallelepipeds are to each other as the products of their three dimensions.

Let the parallelepipeds P and P' have the dimensions a, b, and c and a', b', and c' respectively. Compare P and P' with a third parallelepiped whose dimensions are a, b, and c'.

Proposition 256 Theorem

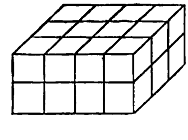
The volume of a rectangular parallelepiped is equal to the product of its three dimensions.

HINT. Find the ratio of the parallelepiped to the cubic unit taken as the unit of volume.

COR. I. *The volume of a rectangular parallelepiped is equal to the product of its base and altitude.*

COR. II. *The volume of a cube is equal to the cube of its edge.* For this reason the *third power* of a quantity is called the *cube* of the quantity. The *volume* of a cube having the line AB as an edge is written $\overline{AB}^3$.

DISCUSSION. When the three dimensions are of such lengths that the linear unit is contained an exact number of times in each, this proposition is rendered evident by dividing the rectangular parallelepiped into cubes, each equal to the unit of volume. For example, if the three dimensions are 4 linear units, 3 linear units, and 2 linear units, the parallelepiped may be divided into 24 cubes, each equal to the unit of volume; that is, the volume is equal to $4 \times 3 \times 2$ units of volume.



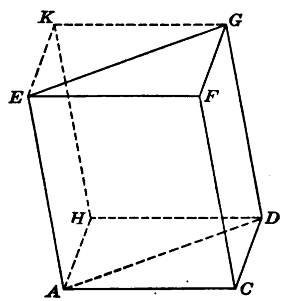
Proposition 257 Theorem

The volume of any parallelepiped is equal to the product of its base and altitude.

Consult Prop. 252 and Prop. 256.

Proposition 258 Theorem

The volume of a triangular prism is equal to the product of its base and altitude.



Hypothesis. Let V denote the volume, B the base, and h the altitude of the triangular prism $ACD\text{-}E$.

Conclusion. $V = B \times h$.

Proof. Construct the parallelepiped $ACDH\text{-}F$, having AC, CD, and CF as edges.

$$ACD\text{-}E = \tfrac{1}{2}\,ACDH\text{-}F. \qquad (?)$$

$$\triangle ACD = \tfrac{1}{2}\,\square\, ACDH. \qquad (?)$$

$$\text{Volume of } ACDH\text{-}F = ACDH \times h. \qquad (?)$$

$$\therefore V = B \times h. \qquad (?)$$

Q. E. D.

Proposition 259 Theorem

The volume of any prism is equal to the product of its base and altitude.

HINT. Let the prism be divided into triangular prisms by the planes determined by a lateral edge and the diagonals of the base drawn from the foot of this edge. Consult Prop. 258 and Ax. 4.

$$V = B \times h.$$

COR. I. *Prisms having equivalent bases and equal altitudes are equivalent.*

COR. II. *Prisms having equivalent bases which lie in parallel planes are equivalent.*

COR. III. *Any two prisms are to each other as the products of their bases and altitudes.*

COR. IV. *Two prisms having equivalent bases are to each other as their altitudes.*

COR. V. *Two prisms having equal altitudes are to each other as their bases.*

Ex. 1084. The volume of an oblique prism is equal to the area of a right section and the length of a lateral edge.

Ex. 1085. The volume of a triangular prism is equal to one half the product of the area of a lateral face and the distance of that face from the opposite lateral edge.

Ex. 1086. The volume of a regular prism is equal to one half the product of the lateral area and the apothem of the base.

Ex. 1087. Show how to divide an oblique prism into two equivalent parts by passing a plane through it parallel to the bases.

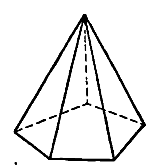
Ex. 1088. Show how to divide a regular prism into two equivalent parts by passing a plane through it parallel to the lateral edges.

PYRAMIDS

A *pyramidal surface* is a surface generated by a straight line which continually moves along a fixed broken line and passes through a fixed point not coplanar with the broken line. The moving line is called the *generatrix*; the broken line is called the *directrix;* and the fixed point is called the *vertex.*

It follows from the definition that a pyramidal surface is composed of planes, the intersections of which are concurrent. The two parts on opposite sides of the vertex are called the *upper* and *lower nappes.*

A *pyramid* is a polyhedron bounded by a closed pyramidal surface and a plane cutting the generatrix in every position. The section of the pyramidal surface formed by the plane is called the *base* of the pyramid, the faces formed on the pyramidal surface are called the *lateral faces,* the intersections of the lateral faces are called the *lateral edges,* and the sum of the areas of the lateral faces is called the *lateral area.*



NOTE. Only pyramids having convex polygons for bases are considered in this book.

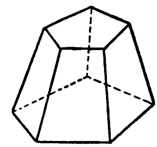
A pyramid is called *triangular, quadrangular, pentagonal,* etc., according as its bases are triangles, quadrilaterals, pentagons, etc.

NOTE. A triangular pyramid is a tetrahedron, and any one of its faces may be taken as its base.

The perpendicular distance from the vertex to the base is called the *altitude* of the pyramid.

A *regular pyramid* is a pyramid having for its base a regular polygon, the centre of which is the foot of the altitude.

A *truncated pyramid* is the part of a pyramid included between the base and a section cutting all the edges. The base of the pyramid and the section thus made are respectively the lower base and the upper base of the truncated pyramid.



A *frustum* of a pyramid is a truncated pyramid whose bases lie in parallel planes. The perpendicular distance between the bases of a frustum is called the *altitude.*

The following principles follow at once from the preceding definitions:

(i) *The lateral edges of a regular pyramid are equal.*

(ii) *The lateral edges of a frustum of a regular pyramid are equal.*

(iii) *The lateral faces of a pyramid are triangles.*

(iv) *The lateral faces of a regular pyramid are congruent isosceles triangles.*

(v) *The altitudes of the lateral faces of a regular pyramid are equal.*

(vi) *The lateral faces of a frustum of a pyramid are trapezoids.*

(vii) *The lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids.*

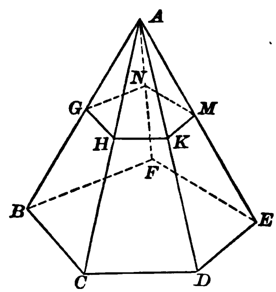
(viii) *The altitudes of the lateral faces of a frustum of a regular pyramid are equal.*

Definitions. The altitude of any lateral face of a regular pyramid is called the *slant height* of the pyramid.

The altitude of any lateral face of a frustum of a regular pyramid is called the *slant height* of the frustum.

Proposition 260 Theorem

The section of a pyramid made by a plane parallel to the base is a polygon similar to the base.



Hypothesis. In the pyramid *A-BCDEF*, *G-M* is a section made by a plane ∥ to the base *B-E*.

Conclusion. *G-M* ∼ *B-E*.

Consult Prop. 214, Prop. 222, and Prop. 79, Cor. III.

COR. *The sections of a pyramid made by two parallel planes which cut all the lateral edges are similar polygons.*

Proposition 261 Theorem

If a pyramid is cut by a plane parallel to its base, the lateral edges and the altitude are divided proportionally.

Consult Prop. 227, Cor.

Ex. 1089. If a plane divides the lateral edges of a pyramid proportionally, it is parallel to the base of the pyramid.

Proposition 262 Theorem

If a pyramid is cut by a plane parallel to its base, the area of the section is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the prism.

Consult Prop. 155 and Prop. 261.

COR. *The areas of two parallel sections of a pyramid are to each other as the squares of their distances from the vertex.*

Proposition 263 Theorem

If two pyramids having equal altitudes are cut by planes parallel to their bases and at equal distances from their vertices, the sections are to each other as their bases.

Consult Prop. 262.

COR. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to their bases and at equal distances from their vertices are equivalent.*

Proposition 264 Theorem

The lateral area of a regular pyramid is equal to one half the product of the perimeter of its base and its slant height.

Consult Prop. 151 and Ax. 4.

Let S denote the lateral area, p the perimeter of the base, and l the slant height of a regular pyramid. Then

$$S = \tfrac{1}{2} p \times l.$$

Ex. 1090. The lateral area of any pyramid is greater than the area of the base.

Proposition 265 Theorem

The lateral area of a frustum of a regular pyramid is equal to one half the product of the sum of the perimeters of its bases and its slant height.

Consult Prop. 152 and Ax. 4.

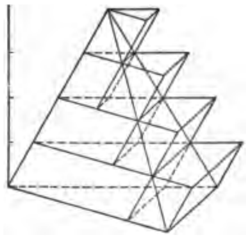
Let S denote the lateral area, p and p' the perimeters of the bases, and l the slant height of a frustum of a regular pyramid. Then

$$S = \tfrac{1}{2}(p + p') \times l.$$

DEFINITIONS. If the altitude of a pyramid is divided into equal parts, and through the points of division planes are passed parallel to the base of the pyramid and on the sections made by these planes as upper bases prisms are constructed having their altitudes equal to one of the equal parts of the altitude of the pyramid and their edges parallel to an edge of the pyramid, the prisms lie wholly within the pyramid and are said to be *inscribed in* the pyramid. Prisms similarly constructed on the sections as lower bases lie partly without the pyramid and are said to be *circumscribed about* the pyramid.

Proposition 266 Theorem

If a series of prisms is inscribed in or circumscribed about a triangular pyramid, the sum of the volumes of the prisms approaches the volume of the pyramid as its limit as the number of prisms is increased indefinitely.



Hypothesis. Let V denote the volume and h the altitude of a triangular pyramid. Let v denote the sum of the volumes of a series of inscribed prisms, and v' the sum of a series of circumscribed prisms, all the prisms having equal altitudes.

Conclusion. As the number of prisms is increased indefinitely, $v \doteq V$ and $v' \doteq V$.

Proof. Each inscribed prism is equivalent to the circumscribed prism directly above it. (?)

Hence, denoting the volume of the lowest circumscribed prism by w, $$v' - v = w.$$

$$v < V < v'. \quad (?)$$

$$\therefore V - v \text{ and } v' - V \text{ are both less than } w. \quad (?)$$

If, now, the number of parts into which h is divided is increased indefinitely, w can be made so small as to become and remain less than any assigned quantity, however small.

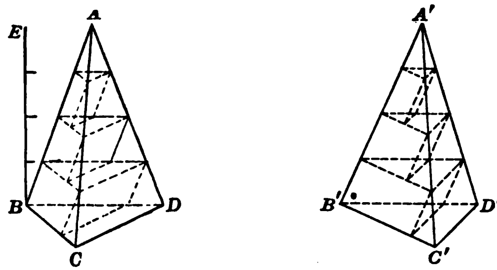
Hence $V - v$ and $v' - V$ can be made so small as to become and remain less than any assigned quantity, however small.

$$\therefore v \doteq V \text{ and } v' \doteq V. \quad (?)$$

Q. E. D.

Proposition 267 Theorem

Triangular pyramids having equivalent bases and equal altitudes are equivalent.



Hypothesis. $A\text{-}BCD$ and $A'\text{-}B'C'D'$ are two triangular pyramids having equivalent bases and equal altitudes. Let their volumes be denoted by V and V' respectively.

Conclusion. $V = V'$.

Proof. Place the pyramids so that their bases shall lie in the same plane, and let BE be the common altitude.

Divide BE into any number of equal parts.

Through the points of division pass planes ∥ to the plane of the bases.

The corresponding sections thus formed are equivalent. (?)

Using these sections as upper bases, construct a series of inscribed prisms in each pyramid.

The corresponding prisms are equivalent. (?)

∴ the sum of the volumes of the prisms inscribed in $A\text{-}BCD$ is equivalent to the sum of the volumes of the prisms inscribed in $A'\text{-}B'C'D'$.

Denoting the total volumes by W and W' respectively,

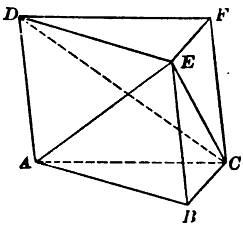
$$W = W'.$$

If, now, the number of parts into which BE is divided is increased indefinitely, $W \doteq V$ and $W' \doteq V$. (?)

To be completed by the student, using the method of limits.]

Proposition 268 Theorem

A triangular pyramid is equivalent to one third of a triangular prism having the same base and the same altitude



Hypothesis. *E-ABC* is a triangular pyramid, and *DEF-ABC* is a triangular prism having the same base and altitude.

Conclusion. Pyramid *E-ABC* = $\frac{1}{3}$ prism *DEF-ABC*.

Proof. The prism is made up of the triangular pyramid *E-ABC* and the quadrangular pyramid *E-ACFD*.

Let the plane determined by *E*, *C*, and *D* divide *E-ACFD* into two triangular pyramids *E-CFD* and *E-ACD*.

$\triangle CFD \equiv \triangle ACD$. (?)

$\therefore$ pyramid *E-CFD* = pyramid *E-ACD*. (?)

Pyramid *E-ABC* = pyramid *C-EDF* (same as *E-CFD*). (?)

$\therefore$ pyramid *E-ABC* is one of three equivalent pyramids which make up the prism *DEF-ABC*.

$\therefore$ pyramid *E-ABC* = $\frac{1}{3}$ prism *DEF-ABC*.

Q. E. D.

COR. *The volume of a triangular pyramid is equal to one-third the product of its base and altitude.*

Let V denote the volume, B the base, and h the altitude of a triangular pyramid. Then

$$V = \frac{1}{3} B \times h.$$

Proposition 269 Theorem

The volume of any pyramid is equal to one third the product of its base and altitude.

HINT. Let the pyramid be divided into triangular pyramids by planes determined by a lateral edge and the diagonals of the base drawn from the foot of this edge. Consult Prop. 268, Cor., and Ax. 4.

$$V = \tfrac{1}{3} B \times h.$$

COR. I. *Pyramids having equivalent bases and equal altitudes are equivalent.*

COR. II. *If the bases of two pyramids are equivalent and lie in the same plane and their vertices lie in a line parallel to the plane of the bases, the two pyramids are equivalent.*

COR. III. *Any two pyramids are to each other as the products of their bases and altitudes.*

COR. IV. *Two pyramids having equivalent bases are to each other as their altitudes.*

COR. V. *Two pyramids having equivalent altitudes are to each other as their bases.*

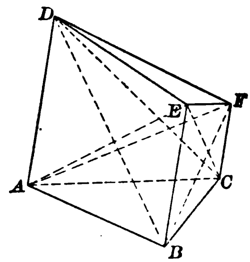
DISCUSSION. The volume of any polyhedron may be found by dividing it into pyramids, and then finding the sum of the volumes of these pyramids. This may be done in several different ways.

One of the simplest methods consists in drawing lines from any point within the polyhedron to all the vertices; the polyhedron is thus divided into pyramids whose bases are the faces of the polyhedron, and whose common vertex is the interior point.

Another method consists in drawing all the diagonals that can be drawn from a single vertex, both within the polyhedron and in the faces which meet at this vertex.

* Proposition 270 Theorem

A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism, and whose vertices are the vertices of the inclined section.



Hypothesis. *ABC-DEF* is a truncated triangular prism whose base is *ABC*, and whose inclined section is *DEF*.

Conclusion. *ABC-DEF* is equivalent to the sum of three pyramids whose common base is *ABC*, and whose vertices are *D*, *E*, and *F*.

Proof. Let the planes determined by *A*, *C*, and *E*, and *C*, *D*, and *E* divide the truncated triangular prism into three pyramids, *E-ABC*, *E-ACD*, and *E-CDF*.

The pyramid *E-ABC* has *ABC* for its base and *E* for its vertex.

Let the pyramid *D-ABC* be cut off by the plane *DBC*.

$E\text{-}ACD = B\text{-}ACD$ (same as $D\text{-}ABC$). (?)

Let the pyramid *F-ABC* be cut off by the plane *FAB*.

$\triangle ACF = \triangle CDF.$ (?)

$E\text{-}CDF = E\text{-}ACF = B\text{-}ACF$ (same as $F\text{-}ABC$). (?)

∴ *ABC-DEF* is equivalent to the sum of the three pyramids *E-ABC*, *D-ABC*, and *F-ABC*, whose common base is *ABC*, and whose vertices are *D*, *E*, and *F*.

Q.E.D.

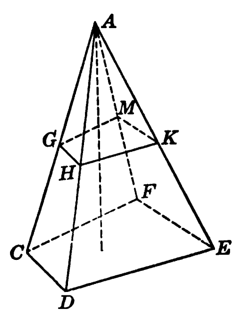
Cor. I. *The volume of a truncated right triangular prism is equal to the product of its base and one-third the sum of its lateral edges.* The lateral edges are the altitudes of the three pyramids whose sum is equivalent to the truncated prism.

Cor. II. *The volume of any truncated triangular prism is equal to the product of a right section and one-third the sum of its lateral edges.* The truncated prism is divided by the right section into two truncated right prisms.

Remark. The volume of a truncated prism with any number of lateral faces can be found by dividing it into truncated triangular prisms.

Proposition 271 Theorem

The volume of a frustum of a pyramid is equal to one third the product of its altitude and the sum of its lower base, its upper base, and the mean proportional between the bases.



Hypothesis. Let V denote the volume, h the altitude, B the lower base, and B' the upper base of C-K, a frustum of the pyramid A-$CDEF$.

Conclusion. $V = \frac{1}{3} h (B + B' + \sqrt{B \times B'})$

Proof. Frustum $C\text{-}K$ = pyramid $A\text{-}CDEF$ – pyramid $A\text{-}GHKM$.

Let h' denote the altitude of pyramid $A\text{-}GHKM$.

$$\text{Then } V = \tfrac{1}{3}B(h + h') - \tfrac{1}{3}B' \times h'.$$
$$= \tfrac{1}{3}h \times B + \tfrac{1}{3}h'(B - B'). \qquad (?)$$

$$\frac{B}{B'} = \frac{(h + h')^2}{h'^2}. \qquad (?)$$

$$\therefore \frac{\sqrt{B}}{\sqrt{B'}} = \frac{h + h'}{h'}. \qquad (?)$$

$$\frac{\sqrt{B} - \sqrt{B'}}{\sqrt{B'}} = \frac{h}{h'}. \qquad (?)$$

$$\therefore h' = \frac{h\sqrt{B'}}{\sqrt{B} - \sqrt{B'}}. \qquad (?)$$

$$\therefore V = \tfrac{1}{3}h \times B + \tfrac{1}{3} \times \frac{h\sqrt{B'}}{\sqrt{B} - \sqrt{B'}}(B - B') \qquad (?)$$

$$= \tfrac{1}{3}h \times B + \tfrac{1}{3}h\sqrt{B'}(\sqrt{B} + \sqrt{B'}) \qquad (?)$$

$$= \tfrac{1}{3}h \times B + \tfrac{1}{3}h\sqrt{B \times B'} + \tfrac{1}{3}h B' \qquad (?)$$

$$= \tfrac{1}{3}h(B + B' + \sqrt{B \times B'}.) \qquad (?)$$

Q.E.D.

COR. *A frustum of a pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.*

Let V denote the volume, B and B' the bases, and h the altitude of a frustum of a pyramid. Then

$$V = \tfrac{1}{3} h (B + B' + \sqrt{B \times B'})$$

THE PRISMATOIDAL FORMULA

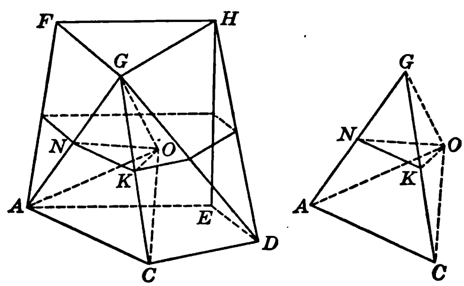
A *prismatoid* is a polyhedron having for bases two polygons in parallel planes, and for lateral faces triangles, trapezoids, or parallelograms with one side in common with one base and the opposite vertex or side in common with the other base.

The *altitude* of a prismatoid is the perpendicular distance between the bases.

The *mid-section* of a prismatoid is the section made by a plane parallel to the bases and midway between them.

Proposition 272 Theorem

The volume of a prismatoid is equal to one sixth the product of the altitude and the sum of the bases and four times the mid-section.



Hypothesis. Let V denote the volume, B and B' the bases, M the mid-section, and h the altitude of a prismatoid.

Conclusion. $V = \frac{1}{6}h\,(B + B' + 4M)$.

Proof. If any lateral face is a trapezoid or a ▱, divide it into two ⚠ by drawing a diagonal.

Take any point O in the mid-section, and pass planes through O and each edge of the prismatoid. The prismatoid is thus divided into pyramids which have O as a common vertex. The bases of these pyramids are B, B', and the $\triangle$s which form the lateral surface of the prismatoid.

Volume of pyramid with base $B = \frac{1}{6}h \times B$. (?)

Volume of pyramid with base $B' = \frac{1}{6}h \times B'$. (?)

The remainder of the prismatoid is made up of triangular pyramids, one of which is $O\text{-}GAC$.

$$\frac{\triangle GAC}{\triangle GNK} = \frac{\overline{AC}^2}{\overline{NK}^2} = 4. \quad (?)$$

$$\therefore \triangle GAC = 4 \triangle GNK. \quad (?)$$

$$\therefore \text{pyramid } O\text{-}GAC = 4 \text{ pyramid } O\text{-}GNK. \quad (?)$$

Volume of $O\text{-}GNK$ (same as $G\text{-}ONK$) $= \frac{1}{6}h \times \triangle ONK$. (?)

$$\therefore \text{volume of } O\text{-}GAC = \frac{4}{6}h \times \triangle ONK. \quad (?)$$

In like manner it can be proved that the volume of every pyramid having its base in the lateral surface is equal to $\frac{4}{6}h$ multiplied by that part of the mid-section which it includes.

Hence the total volume of the pyramids whose bases form the lateral surface is $\frac{4}{6}h \times M$. (?)

$$\therefore V = \frac{1}{6}h \times B + \frac{1}{6}h \times B' + \frac{4}{6}h \times M.$$

$$= \frac{1}{6}h\,(B + B' + 4M). \quad (?)$$

Q. E. D.

DISCUSSION. The expression

$$V = \frac{1}{6}h\,(B + B' + 4M)$$

is known as the *prismatoidal formula*.

This formula is used extensively in practical work, and it will be shown that it can be used to find the volume of any solid studied by the ordinary student of Solid Geometry.

DEFINITION. A *wedge* is a prismatoid whose lower base is a rectangle and whose upper base is a line parallel to an edge of the lower base.

Ex. 1091. By making the proper substitutions in the prismatoidal formula, derive the formula for a prism, as given on page 333.

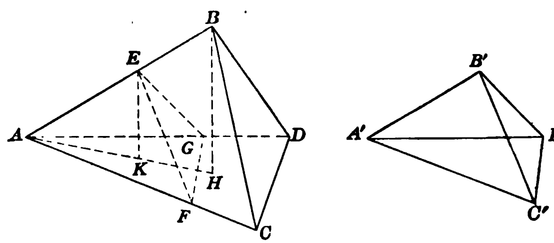
Ex. 1092. By making the proper substitutions in the prismatoidal formula, derive the formula for a pyramid, as given on page 342.

Ex. 1093. By making the proper substitutions in the prismatoidal formula, derive the formula for a frustum of a pyramid, as given on page 345.

Ex. 1094. Let V denote the volume, b the length of the base, a the width of the base, b' the length of the upper edge, and h the altitude of a wedge. Show that $V = \frac{1}{6} ha (2b + b')$.

Proposition 273 Theorem

Two tetrahedrons having a trihedral angle of one congruent to a trihedral angle of the other are to each other as the products of the edges including the equal trihedral angles.



Hypothesis. In the tetrahedrons $A\text{-}BCD$ and $A'\text{-}B'C'D'$, trihedral $\angle A\text{-}BCD \equiv$ trihedral $\angle A'\text{-}B'C'D'$. Let their volumes be denoted by V and V' respectively.

Conclusion. $\dfrac{V}{V'} = \dfrac{AB \times AC \times AD}{A'B' \times A'C' \times A'D'}$.

Proof. Place the tetrahedron $A'\text{-}B'C'D'$ in the position $A\text{-}EFG$, the trihedral $\angle A'\text{-}B'C'D'$ coinciding with the trihedral $\angle A\text{-}BCD$.

Draw BH and $EK \perp$ to ACD.

Then A, K, and H lie in the same straight line. (?)

$$\frac{V}{V'} = \frac{\triangle ACD \times BH}{\triangle AFG \times EK} = \frac{\triangle ACD}{\triangle AFG} \times \frac{BH}{EK}. \quad (?)$$

$$\frac{\triangle ACD}{\triangle AFG} = \frac{AC \times AD}{AF \times AG}. \quad (?)$$

$$\frac{BH}{EK} = \frac{AB}{AE}. \quad (?)$$

$$\therefore \frac{V}{V'} = \frac{AC \times AD}{AF \times AG} \times \frac{AB}{AE} = \frac{AB \times AC \times AD}{AE \times AF \times AG}$$

$$= \frac{AB \times AC \times AD}{A'B' \times A'C' \times A'D'}. \quad (?)$$

Q. E. D.

SIMILAR POLYHEDRONS

Two polyhedrons are *similar* when they have the same number of faces, similar each to each and similarly placed, and their homologous polyhedral angles are congruent.

The following principles follow at once from the above definition and the principles of similar polygons proved in Plane Geometry.

(i) *Homologous edges of similar polyhedrons are proportional.*

(ii) *Any two homologous lines in two similar polyhedrons are in the same ratio as any two homologous edges.*

(iii) *Homologous faces of similar polyhedrons are to each other as the squares of any two homologous lines.*

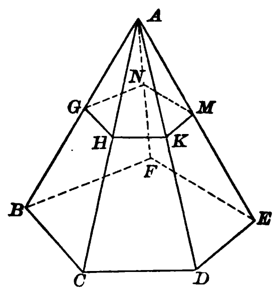
(iv) *The total surfaces of any two similar polyhedrons are to each other as the squares of any two homologous lines.*

Definition. In two similar polyhedrons, the ratio of any two homologous edges is called the *ratio of similitude* of the polyhedrons.

Note. Similar polygons have not only the same number of faces, but also the same number of vertices and the same number of edges.

Proposition 274 Theorem

If a pyramid is cut by a plane parallel to the base, the pyramid cut off is similar to the original pyramid.



Hypothesis. The pyramid *A-BCDEF* is cut by the plane *G-M*, which is ∥ to the base *B-E*.

Conclusion. Pyramid *A-GHKMN* ~ pyramid *A-BCDEF*.

Proof. Polygon *GHKMN* ~ polygon *BCDEF*. (?)

$\triangle AGH \sim \triangle ABC$. (?)

In like manner it can be proved that $\triangle AHK \sim \triangle ACD$, $\triangle AKM \sim \triangle ACD$, etc.

Polyhedral $\angle A$ is common to the two pyramids.

$\angle AGH = \angle ABC$, $\angle AGN = \angle ABF$,
and $\angle HGN = \angle CBF$. (?)

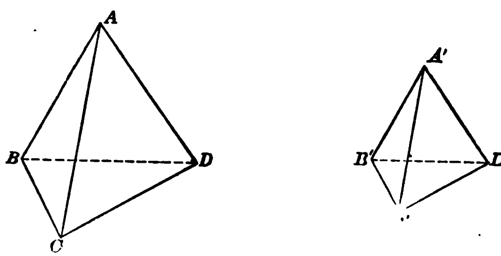
$\therefore$ trihedral $\angle G - AHN \equiv$ trihedral $\angle B\text{-}ACF$. (?)

In like manner it can be proved that trihedral $\angle H\text{-}AGK \equiv$ trihedral $\angle C\text{-}ABD$, trihedral $\angle K\text{-}AHM \equiv$ trihedral $\angle D\text{-}ACE$, etc.

$\therefore$ pyramid $A\text{-}GHKMN \sim$ pyramid $A\text{-}BCDEF$. (?)

Proposition 275 Theorem

Two tetrahedrons are similar when the faces including a trihedral angle of one are similar respectively to the faces including a trihedral angle of the other, and are similarly placed.



Hypothesis. In the tetrahedrons $A\text{-}BCD$ and $A'\text{-}B'C'D'$, $\triangle$s ABC, ACD, and ABD are similar respectively to $\triangle$s $A'B'C'$, $A'C'D'$, and $A'B'D'$.

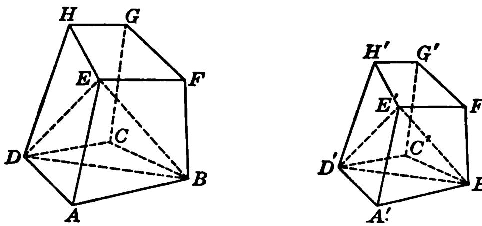
Conclusion. Tetrahedron $A\text{-}BCD \sim$ tetrahedron $A'\text{-}B'C'D'$.

HINT. Prove that $\triangle BCD \sim \triangle B'C'D'$, and that the homologous trihedral $\angle$s are congruent.

Ex. 1095. Two tetrahedrons are similar when a dihedral angle of one is equal to a dihedral angle of the other, and the faces including these angles are similar each to each, and are similarly placed.

Proposition 276 Theorem

Two similar polyhedrons can be divided into the same number of tetrahedrons, similar each to each, and similarly placed.



Hypothesis. AG and $A'G'$ are similar polyhedrons in which E and E' are homologous vertices.

Conclusion. AG and $A'G'$ can be divided into the same number of tetrahedrons, similar each to each, and similarly placed.

Proof. In all the faces of AG, except those about E, draw diagonals dividing these faces into $\triangle$s, and pass planes through E and these diagonals.

In like manner draw homologous diagonals in the faces of $A'G'$, except those about E', and pass planes through E', and these diagonals.

The two polyhedrons are thus divided into the same number of tetrahedrons.

Let $A\text{-}BDE$ and $A'\text{-}B'D'E'$ be two tetrahedrons similarly situated.

$$\triangle ABD \sim \triangle A'B'D',\ \triangle ABE \sim \triangle A'B'E',$$
$$\text{and } \triangle ADE \sim \triangle A'D'E'. \qquad (?)$$
$$\text{trihedral } \angle A\text{-}BDE \equiv \text{trihedral } \angle A'\text{-}B'D'E'. \qquad (?)$$
$$\therefore \text{ tetrahedron } A\text{-}BDE \sim \text{tetrahedron } A'\text{-}B'D'E'. \qquad (?)$$

Removing these two tetrahedrons, the remaining polyhedrons are similar, since removing similar $\triangle$s from similar faces leaves similar faces, and removing congruent trihedral ∠s from congruent polyhedral ∠s leaves congruent polyhedral ∠s.

Continuing this process, AG and $A'G'$ can be divided into the same number of tetrahedrons, similar each to each, and similarly placed.

Q. E. D.

Ex. 1096. State and prove the converse of Prop. 276.

Ex. 1097. If the ratio of similitude of two similar polyhedrons is unity, the polyhedrons are congruent.

Ex. 1098. If the homologous faces of two similar tetrahedrons are respectively parallel, the lines which join their homologous vertices intersect at a common point.

Proposition 277 Theorem

Similar tetrahedrons are to each other as the cubes of their homologous edges.

Consult Prop. 273

Proposition 278 Theorem

Similar polyhedrons are to each other as the cubes of their homologous edges.

Select two homologous vertices and draw diagonals in all the faces except those about the selected vertices. Pass planes through these vertices and the diagonals, thus dividing the polyhedrons into similar tetrahedrons.

COR. *Similar polyhedrons are to each other as the cubes of any two homologous lines.*

REGULAR POLYHEDRONS

A *regular polyhedron* is a polyhedron whose faces are congruent regular polygons, and whose polyhedral angles are congruent.

Proposition 279 Problem

To determine how many regular convex polyhedrons are possible.

A convex polyhedral ∠ must have at least three face ∠s, and the sum of these face ∠s is less than 360°. (?)

1. Suppose the faces of the regular polyhedron to be equilateral △s, each ∠ of which is 60°. A convex polyhedral ∠ can be formed by combining three, four, or five equilateral △s, for three, four, or five times 60° is less than 360°. Not more than five equilateral △s can be combined to form a convex polyhedral ∠, for six times 60° is 360°.

2. Suppose the faces of the regular polyhedron to be squares, each ∠ of which is 90°. A convex polyhedral ∠ can be formed by combining three squares, for three times 90° is less than 360°. Not more than three squares can be combined to form a convex polyhedral ∠, for four times 90° is 360°.

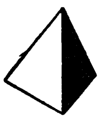
3. Suppose the faces of the regular polyhedron to be regular pentagons, each angle of which is 108°. A convex polyhedral ∠ can be formed by combining three regular pentagons, for three times 108° is less than 360°. Not more than three regular pentagons can be combined to form a convex polyhedral ∠, for four times 108° is more than 360°.

4. Since each ∠ of a regular hexagon is 120°, the sum of three such ∠s is 360°, and no convex polyhedral ∠ can be formed by combining regular hexagons. In like manner it may be proved that no convex polyhedral ∠ can be

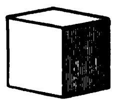
formed by combining regular polygons of more than six sides.

Accordingly, only five regular polygons are possible.

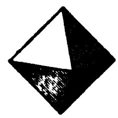
Q. E. F.



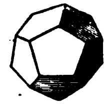
Tetrahedron



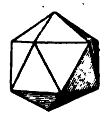
Hexahedron



Octahedron

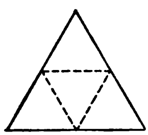


Dodecahedron

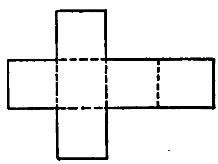


Icosahedron

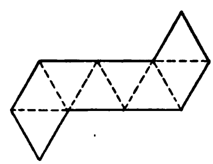
To construct cardboard models of the regular polyhedrons, cut out figures of the shapes given below and cut half through along the dotted lines. The models can then be bent into the desired forms and held together by pasting strips of paper along the edges.



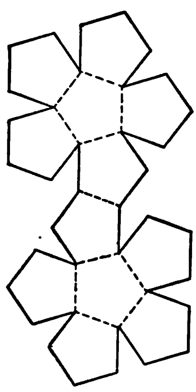
Tetrahedron



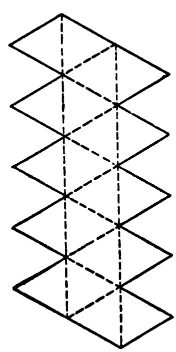
Hexahedron



Octahedron



Dodecahedron



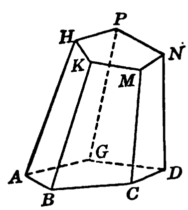
Icosahedron

The following table shows the properties of the five regular polyhedrons:

NAME	No. of faces	No. of sides in each face	No. of vertices	No. of face angles at each vertex	No. of edges
Tetrahedron	4	3	4	3	6
Hexahedron (cube)	6	4	8	3	12
Octahedron	8	3	6	4	12
Dodecahedron	12	5	20	3	30
Icosahedron	20	3	12	5	30

*Proposition 280 Theorem

The number of edges of any polyhedron increased by two is equal to the number of its vertices increased by the number of its faces.



Hypothesis. In the polyhedron AN, let E denote the number of edges, V the number of vertices, and F the number of faces.

Conclusion. $E + 2 = V + F$.

Proof. In any one face, as $ABCDG$, $E = V$.

If a second face $ABKH$ is annexed by joining one of its edges to an edge of the first face, a surface is formed having one edge and two vertices common to the two faces. That is, the number of edges is one more than the number of vertices.

$\therefore$ for two faces, $E = V + 1$.

For each additional face the number of new edges is one more than the number of new vertices.

$\therefore$ for $F - 1$ faces, $E = V + F - 2$

When the last face is added, E and V remain the same.

$\therefore$ for F faces, $E = V + F - 2$,

or $E + 2 = V + F$.

Q. E. D.

REMARK. This theorem is commonly known as *Euler's Theorem*.

*Proposition 281 Theorem

The sum of the face angles of any polyhedron is equal to four right angles taken as many times as the polyhedron has vertices less two.

Hypothesis. Let S denote the sum of the face $\measuredangle$s, and V the number of vertices of any polyhedron.

Conclusion. $S = (V - 2)\ 4$ rt. $\measuredangle$s.

Proof. Let E denote the number of edges, and F the number of faces.

Since each edge is common to two faces, $2E$ denotes the number of edges of all its faces considered as independent polygons.

If the edges of each polygon are produced so as to form an exterior $\angle$ at each vertex,

$S +$ the sum of the exterior $\measuredangle$s $= 2E \times 2$ rt. $\measuredangle$s $= 4E$ rt. $\measuredangle$s. (?)

The sum of the exterior $\measuredangle$s $= F \times 4$ rt. $\measuredangle$s. (?)

$\therefore S = (E - F)\ 4$ rt. $\measuredangle$s. (?)

But $E + 2 = V + F$, and $E - F = V - 2$. (?)

$\therefore S = (V - 2)\ 4$ rt. $\measuredangle$s. (?)

Q. E. D.

EXERCISES

Ex. 1099. If a right section of a prismatic surface is an equilateral triangle, every section which has one of its sides parallel to one of the sides of the given section is an isosceles triangle.

Ex. 1100. The dimensions of a rectangular parallelepiped are a, b, c. Prove that the difference between the area of a square whose side is $a + b + c$ and the area of the surface of the parallelepiped is the square of the diagonal of the latter.

Ex. 1101. Any line drawn through the point of intersection of the diagonals of a parallelepiped and terminated by two opposite faces is bisected at that point.

Ex. 1102. In any quadrangular prism, the two diagonals which connect either pair of opposite lateral edges bisect each other.

Ex. 1103. If the diagonals of a parallelepiped are equal, the parallelepiped is rectangular.

Ex. 1104. The perpendicular drawn from the mid-point of a diagonal of a rectangular parallelepiped to a lateral edge bisects the edge, and is equal to one half the projection of the diagonal upon the base.

Ex. 1105. The sum of two opposite lateral edges of a truncated parallelepiped is equal to the sum of the other two lateral edges.

Ex. 1106. The volume of a truncated parallelepiped is equal to the product of the area of a right section and one fourth the sum of the lateral edges.

Ex. 1107. The perpendicular drawn to the lower base of a truncated right triangular prism from the point of intersection of the medians of the upper base is equal to one third the sum of the lateral edges.

Ex. 1108. The volume of a truncated triangular prism is equal to the product of the area of its lower base and the perpendicular drawn to the lower base from the point of intersection of the medians of the upper base.

Ex. 1109. Show how to construct a parallelepiped having three of its edges on three indefinite skew lines.

Ex. 1110. Prove that the product of the lengths of the common perpendiculars of the three lines in Ex. 1109 is in general less than the volume of the parallelepiped. Is this product ever equal to the volume?

Ex. 1111. The volume of a regular pyramid is equal to one third the product of its lateral area and the distance from the centre of the base to any lateral face.

Ex. 1112. If lines are drawn joining the mid-points of two pairs of opposite edges of a triangular pyramid, the figure thus formed is a parallelogram.

Ex. 1113. A section of a tetrahedron made by a plane parallel to two opposite edges is a parallelogram.

Ex. 1114. Two tetrahedrons are congruent when two faces and the included dihedral angle of one are equal respectively to two faces and the included dihedral angle of the other, and the equal parts are similarly placed.

Ex. 1115. Two tetrahedrons are congruent when three faces of one are equal respectively to three faces of the other and are similarly placed.

Ex. 1116. The plane which bisects a dihedral angle of a tetrahedron divides the opposite edge into segments which are proportional to the areas of the faces including the dihedral angle.

Ex. 1117. In any tetrahedron, the planes passing through the three lateral edges and the mid-points of the sides of the base intersect in a straight line.

Ex. 1118. The four lines, each joining the vertex of a tetrahedron with the point of intersection of the medians of the opposite face, intersect at a point which divides each of these lines in the ratio 3 : 1.

NOTE. The point of intersection of these lines is called the *centre of gravity* of the tetrahedron.

Ex. 1119. The three lines joining the mid-points of the opposite edges of a tetrahedron are concurrent and mutually bisect each other.

Ex. 1120. In the tetrahedron $A\text{-}BCD$, perpendiculars are drawn from A and B to the opposite faces. If these perpendiculars intersect, prove that AB is perpendicular to CD.

Ex. 1121. If in the tetrahedron $A\text{-}BCD$, the edges AB and CD are perpendicular to each other, prove that perpendiculars drawn from A and B to the opposite faces intersect.

Ex. 1122. In the tetrahedron $A\text{-}BCD$, E and F are the feet of the perpendiculars drawn from a point within the tetrahedron to the faces ABD and ABC respectively; prove that $\overline{AE}^2 - \overline{BE}^2 = \overline{AF}^2 - \overline{BF}^2$.

Ex. 1123. If in the tetrahedron A-BCD the face angles at A are right angles and AE is the altitude upon the face BCD, prove that

$$\frac{1}{\overline{AB}^2} + \frac{1}{\overline{AC}^2} + \frac{1}{\overline{AD}^2} = \frac{1}{\overline{AE}^2}.$$

Ex. 1124. If in the tetrahedron A-BCD the face angles at A are right angles, prove that the square of the area of the triangle BCD is equivalent to the sum of the squares of the areas of the other three faces.

Ex. 1125. If a perpendicular is drawn from either vertex of a regular tetrahedron to the opposite face, the foot of the perpendicular divides each median of that face in the ratio 2 : 1.

Ex. 1126. The four altitudes of a regular tetrahedron are equal.

Ex. 1127. Three times the square of the altitude of a regular tetrahedron is equal to twice the square of an edge.

Ex. 1128. The line joining the mid-points of two opposite edges of a regular tetrahedron is perpendicular to these edges.

Ex. 1129. The square of the line joining the mid-points of two opposite edges of a regular tetrahedron is equal to one half the square of an edge.

Ex. 1130. Any two non-intersecting edges of a regular tetrahedron are perpendicular to each other.

Ex. 1131. The perpendicular drawn from a vertex of a regular tetrahedron to the opposite face is three times the perpendicular drawn from its own foot to any other face.

Ex. 1132. If three faces of a tetrahedron are congruent, the sum of the distances from any point in the fourth face to the other three faces is constant.

Ex. 1133. The sum of the distances from any point within a regular tetrahedron to the four faces is equal to the altitude of the tetrahedron.

Ex. 1134. Any two opposite faces of a regular octahedron lie in parallel planes.

Ex. 1135. The section of a regular octahedron made by a plane passing through the mid-point of an edge and parallel to either face is a regular hexagon.

Ex. 1136. The mid-points of the edges of a regular tetrahedron are at the vertices of a regular octahedron.

PROBLEMS OF COMPUTATION

1. What is the length of the longest line that can be drawn within a rectangular parallelepiped 12 ft. long, 4 ft. wide, and 3 ft. thick?

2. Find the total area of a rectangular parallelepiped whose dimensions are 14 in., 11 in., and $8\frac{1}{2}$ in.

3. If the diagonal of a cube is 24 in., what is the total area of the cube? What is the volume?

4. The area of the entire surface of a cube is S. Find the length of an edge, the length of a diagonal, and the volume.

5. Find the number of square inches on the surface of a cube whose volume is $42\frac{7}{8}$ cu. in.

6. If 180 sq. ft. of lead are required for lining the bottom and sides of a cubical vessel, how many cubic feet of water will it hold?

7. The number of square feet on the surface of a certain cube is equal to the number of cubic feet in its volume. Find the length of an edge.

8. A rectangular parallelepiped has a square base whose side is 23 in., and the diagonal of the parallelepiped is 52 in. Find the volume.

9. The dimensions of the base of a rectangular parallelepiped are 12 ft. and 4 ft., and the number of square feet in the total area is equal to the number of cubic feet in the volume. Find the altitude.

10. In a rectangular parallelepiped whose base is a square, the altitude is 8 in. and the total area is 160 sq. in. Find the volume.

11. The three external dimensions of a box with cover are 2 ft. 8 in., 1 ft. 10 in., and 1 ft. 6 in., and it is constructed of a material 1 in. thick. Find the number of cubic inches of material used.

12. A cube whose edge measures 15 in. is equivalent to a rectangular parallelepiped, two of whose dimensions are 25 in. and 10 in. Find the third dimension and the total area of the parallelepiped.

13. Three edges of a parallelepiped are AB, AC, and AD. Find the ratio of the volumes of the two solids into which the parallelepiped is divided by the plane BCD. Find the ratio of the two solids into which the parallelepiped is divided by the plane bisecting these three edges.

14. The base of a right prism is a triangle whose sides are 12 in., 15 in., and 17 in., and the altitude is $8\frac{1}{2}$ in. Find the lateral area.

15. Find the volume of a triangular prism, the sides of whose base are 8 in., 6 in., and 5 in., and whose altitude is 10 in.

16. The volume of a right prism is 480 cu. in., and its base is an isosceles triangle whose sides are 13 in., 13 in., and 10 in. Find the altitude of the prism.

17. The lateral area and the volume of a regular triangular prism are 144 sq. in. and $96\sqrt{3}$ cu. in. respectively. Find the altitude and an edge of the base.

18. A right prism 16 in. high has for its base a regular hexagon whose side is 12 in. Find the total area.

19. Find the volume of a regular hexagonal prism whose altitude is $6\frac{1}{2}$ ft., and each side of whose base is 4 ft.

20. Find the number of cubic feet of concrete in a dam 250 ft. long, 31 ft. high, 33 ft. wide at the bottom, and 5 ft. wide at the top.

21. The base of a truncated right prism is an isosceles triangle whose sides are 5 in., 5 in., and 6 in., and the lateral edges are 10 in., 13 in., and 16 in. Find the volume.

22. A right section of a truncated prism is an equilateral triangle whose perimeter is $16\frac{1}{2}$ in., and the lateral edges are 8 in., 10 in., and $10\frac{1}{2}$ in. Find the volume.

23. The base of a truncated right prism is a square, each side of which is 6 in., and the lateral edges are 5 in., 8 in., 13 in., and 10 in. Find the volume.

24. Find the volume of a truncated triangular prism if its base contains 40 sq. in., the three lateral edges are 5 in., 10 in., and 20 in., and the projection upon the base of the edge of length 5 in. equals 4 in.

25. Find the lateral edge, lateral area, and volume of a regular pyramid whose altitude is 10 in., and whose base is an equilateral triangle each of whose sides is 8 in.

26. Find the lateral area and volume of a regular pyramid whose slant height is 12 ft., and whose base is an equilateral triangle each of whose sides is $5\sqrt{3}$ ft.

27. Find the lateral edge, lateral area, and volume of a regular quadrangular pyramid, each side of whose base is 10 in., and whose altitude is 12 in.

28. Find the lateral area and volume of a regular hexagonal pyramid, each side of whose base is 4 in., and whose altitude is 5 in.

29. The volume of a pyramid is 210 cu. in., and its base is a triangle whose sides are 5 in., 12 in., and 13 in. Find the altitude of the pyramid.

30. The lateral surface of a regular quadrangular pyramid is composed of four equilateral triangles, and its altitude is 15 in. Find the area of the base.

31. A regular pyramid has a lateral edge of 101 ft. and a square base 40 ft. on a side. Find the volume.

32. The lateral area of a regular pyramid is S, and the base is a square whose edge is a. Find the altitude of the pyramid.

33. Find the volume of a regular hexagonal pyramid whose altitude and slant height are 24 in. and 25 in. respectively.

34. The altitude of a regular hexagonal pyramid is 15 ft., and the apothem of the base is 8 ft. Find the lateral area of the pyramid.

35. The altitude of a regular hexagonal pyramid is 5 ft., and an edge of the base is 4 ft. Find the lateral area of the pyramid.

36. The base of a regular pyramid is a hexagon of side 10 in. The lateral edge is 20 in. Find the volume.

37. The base of a regular pyramid is a regular hexagon which can be inscribed in a circle of radius 10 in. A lateral edge of the pyramid is 20 in. Find the volume.

38. The base of a regular pyramid is a hexagon whose side is 4 ft., and its lateral area is six times the area of the base. Find the altitude of the pyramid.

39. The spire of a church is a regular hexagonal pyramid; each side of the base is 10 ft., and the height is 50 ft. There is also a hollow part which is also a regular hexagonal pyramid; the height of the hollow part is 45 ft., and each side of the base is 9 ft. Find the number of cubic feet of stone in the tower.

40. One lateral edge of a pyramid is 10 in., and the angle formed by this edge and the plane of the base is 45°; the area of the base is 120 sq. in. Find the volume.

41. A regular prism and a regular pyramid have equivalent volumes and equivalent bases. If the altitude of the prism is $7\frac{1}{2}$ ft., what is the altitude of the pyramid?

42. The areas of two mutually perpendicular faces of a tetrahedron are 4 sq. in. and 3 sq. in. The edge between these two faces is 2 in. long. Find the volume of the tetrahedron.

43. The vertices of a tetrahedron are four vertices of a cube, no two of which lie on the same edge of the cube. Find the ratio of the volume of the tetrahedron to the volume of the cube.

44. A pedestal for a statue is in the form of the frustum of a regular quadrangular pyramid. Each side of the lower base is 5 ft., each side of the upper base is 3 ft., and the height is 4 ft. Find the lateral area of the pedestal.

45. The lower base of a frustum of a right pyramid is a square 4 in. on a side. A side of the upper base is half that of the lower base, and the altitude of the frustum is the same as a side of the upper base. Find the volume of the frustum.

46. Find the lateral area and volume of a frustum of a regular quadrangular pyramid, the sides of whose bases are 20 ft. and 4 ft., and whose altitude is 15 ft.

47. The slant height of the frustum of a regular quadrangular pyramid is 37 in., and the sides of the lower base and upper base are 42 in. and 18 in. respectively. Find the volume.

48. Find the volume of the frustum of a triangular pyramid whose bases are 90 sq. in. and 40 sq. in., and whose altitude is 3 in.

49. The volume of a frustum of a pyramid is 84 cu. in., and the bases are squares whose sides are 2 in. and 4 in. Find the altitude.

50. If the bases of a frustum of a pyramid are two regular hexagons whose sides are 1 ft. and 2 ft., and the volume of the frustum is 12 cu. ft., find the altitude.

51. A stick of timber 32 ft. long and 18 in. wide is 15 in. thick at one end and 12 in. thick at the other. Find the number of cubic feet it contains.

52. The volume of a frustum of a pyramid is 38 cu. ft., its altitude is 6 ft., and the area of one base is 9 sq. ft. Find the area of the other base.

53. Find the volume of the frustum of an oblique pyramid from following data: the lower base is a square 4 in. on a side, the upper base is a square 2 in. on a side, and one of the inclined edges, which is 8 in. long, has as its projection upon the lower base one of the diagonals of that base.

54. Find the total surface and volume of a regular tetrahedron whose edge is 8 in.

55. Find the volume of a regular tetrahedron whose altitude is 7 in.

56. The volume of a regular tetrahedron is $\frac{16}{3}\sqrt{2}$ cu. in. Find the edge, slant height, and altitude.

57. Find the length of an edge of a regular tetrahedron whose volume is $18\sqrt{2}$ cu. in.

58. Find the volume of a regular tetrahedron whose total area is $144\sqrt{3}$ sq. in.

59. The vertices of one regular tetrahedron are at the centers of the faces of another regular tetrahedron. Find the ratio of the volumes.

60. Find the sum of the face angles of the polyhedral angle at any vertex of a regular octahedron.

61. The area of one face of a regular octahedron is one square foot. Find the volume.

62. Find the volume of a regular octahedron whose total area is $72\sqrt{3}$ sq. in.

63. Find the ratio of the volume of a cube to that of the regular octahedron whose vertices are the centres of the faces of the cube.

64. The corner of a cube is cut off by a plane passing through the mid-points of the edges which terminate at that vertex, and the process is repeated for each corner of the cube. Find the ratio of the volume of the solid that remains to the volume of the cube.

65. AB, AC, and AD are three edges of a cube which meet in the vertex A. A plane is passed through the mid-points of these edges. If the cube contains 8 cu. ft., find the volume of the corner cut off by the plane and the length of the perpendicular drawn from the centre of the cube to the plane.

66. A pyramid 12 ft. high has a base containing 225 sq. ft. What is the area of a section parallel to the base whose distance from the vertex is 8 ft.?

67. A pyramid whose base is a square 5 in. on an edge is cut by a plane parallel to the base and bisecting the altitude. Find the area of the section.

68. A pyramid 9 ft. high has a square base measuring 6 ft. on a side, and a section parallel to the base measures 2 ft. on a side. Find the distance from the vertex to this section.

69. A section of a pyramid made by a plane parallel to the base is equal to one-ninth of the base in area, and the altitude of the pyramid is 6 ft. How far from the vertex is the plane of the section?

70. At what distance from the vertex must a pyramid whose altitude is 12 in. be cut by a plane parallel to the base in order to divide it into two equivalent parts?

71. The length of a lateral edge of a pyramid is a. At what distances from the vertex will this edge be cut by two planes parallel to the base, which divide the pyramid into three equivalent parts?

72. A regular triangular pyramid has a for its altitude and for each side of its base. Find the area of a section parallel to the base and distance $\frac{1}{2}a$ from the vertex; also find the volume of the pyramid.

73. The stone cap of a gate post is in the form of a regular quadrangular pyramid whose base measures 4 in. on a side and whose altitude is 15 in. If the top of the cap is cut off by a plane parallel to the base and 5 in. above it, what is the volume of the piece cut off?

74. The altitude of a frustum of a pyramid is 16 in., and two homologous sides of its bases are 12 in. and 10 in. Find the ratio of the volume of the frustum to the volume of the entire pyramid.

75. Two similar pyramids have altitudes of 6 ft. and 8 ft. Find the ratio of their surfaces, and also of their volumes.

76. The height of the Great Pyramid is 489 ft. An exact model of the pyramid is made of height 4.89 ft., its side faces being triangles similar to the side faces of the pyramid. What is the ratio of the total lateral area of the pyramid to that of the model?

77. The volumes of two similar polyhedrons are respectively 3 cu. in. and 24 cu. in., and one edge of the first is 5 in. What is the homologous edge of the second?

78. The volumes of two similar polyhedrons are respectively 64 cu. ft. and 216 cu. ft. If the total area of the first polyhedron is 112 sq. ft., what is the total area of the second?

79. The base of a right pyramid is a regular hexagon. The altitude is h and the length of an edge of the base is a. Find the area of the base of a similar pyramid having one eighth of the volume.

80. The altitude of a certain solid is 2 in., its lateral area is 15 sq. in., and its volume is 4 cu. in. Find the altitude and lateral area of a similar solid whose volume is 256 cu. in.

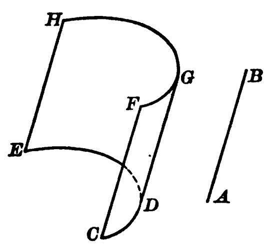
Book VIII

THE CYLINDER AND THE CONE

THE CYLINDER

A *cylindrical surface* is a curved surface generated by a straight line which continually moves along a fixed curve and is constantly parallel to a given straight line not coplanar with the curve.

For example, if the straight line CF moves along the curve CDE so that in every position it is parallel to AB, the surface thus generated is a cylindrical surface.



The moving line is called the *generatrix*, and the curve is called the *directrix*. The generatrix in any position is called an *element* of the surface.

A *cylinder* is a solid bounded by a closed cylindrical surface and two parallel planes. The sections of the cylindrical surface formed by the parallel planes are called the *bases* of the cylinder, and the cylindrical surface is called the *lateral surface*. The area of the lateral surface is called the *lateral area* of cylinder. An element of the cylindrical surface is called an *element* of the cylinder.

A cylinder whose elements are perpendicular to its bases is called a *right* cylinder. A cylinder whose elements are oblique to its bases is called an *oblique* cylinder.

A cylinder whose bases are circles is called a *circular* cylinder. The line joining the centres of the bases is called the *axis* of the circular cylinder.

The perpendicular distance between the bases is called the *altitude* of the cylinder.

A section of a cylinder made by a plane perpendicular to the elements is called a *right section.*

A plane is *tangent* to a cylinder when it passes through an element, but does not meet it again, however far it is produced. The element through which the plane passes is called the *element of contact.*

The following principles follow at once from the preceding definitions:

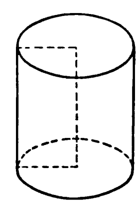
(i) *The elements of a cylinder are parallel and equal.*

(ii) *Any element of a right cylinder is equal to the altitude.*

(iii) *A line drawn through any point in a cylindrical surface parallel to an element is an element.*

Proposition 282 Theorem

A right circular cylinder may be generated by the revolution of a rectangle about one of its sides as an axis.



Prove that the solid generated fulfils *all* the requisites of a right circular cylinder. Consult Prop. 210 and Prop. 225.

DEFINITIONS. On account of the manner in which a right circular cylinder may be generated, it is also called a *cylinder of revolution.* The radius of the base is called the *radius* of the cylinder.

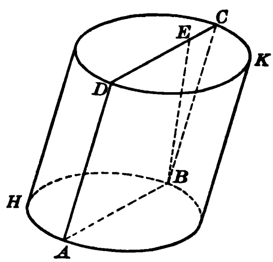
Similar cylinders of revolution are cylinders generated by the revolution of similar rectangles about homologous sides as axes.

A cylinder of revolution whose section through the axis is a square is called an *equilateral* cylinder.

Ex. 1137. The axis of a circular cylinder is equal and parallel to all the elements.

Proposition 283 Theorem

Every section of a cylinder made by a plane passing through an element is a parallelogram.



Hypothesis. *ABCD* is a section of the cylinder *HK*, made by a plane passing through the element *AD*.

Conclusion. *ABCD* is a ▱.

Proof. Through *B* draw *BE* in the plane *AC* ∥ to *AD*.

Then *BE* is an element of the cylinder. (?)

∴ *BE* is the intersection of the plane and the cylindrical surface.

Hence *BE* coincides with *BC*.

∴ *BC* is a straight line ∥ to *AD*. (?)

AB is ∥ to *DC*. (?)

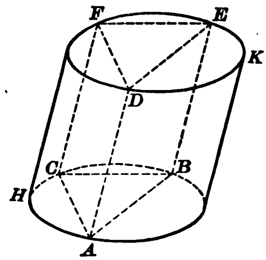
∴ *ABCD* is a ▱. (?)

Q. E. D.

Cor. *Every section of a right cylinder made by a plane passing through an element is a rectangle.*

Proposition 284 Theorem

The bases of a cylinder are congruent.



Hypothesis. ABC and DEF are the bases of the cylinder HK.

Conclusion. Base $ABC \equiv$ base DEF.

Proof. Let A, B, and C be any three points in the perimeter of base ABC, and let the elements passing through A, B, and C be AD, BE, and CF respectively. Draw AB, AC, BC, DE, DF, and EF.

AD is equal and $\parallel$ to BE. (?)

$\therefore AB = DE$. (?)

In like manner it may be proved that $AC = DF$ and $BC = EF$.

$\therefore \triangle ABC \equiv \triangle DEF$. (?)

Apply base ABC to base DEF so that $\triangle ABC$ shall coincide with $\triangle DEF$. Then A, B, and C will fall on D, E, and F respectively.

Since A, B, and C are any three points in the perimeter of base ABC, every point in the perimeter of base ABC will fall on the perimeter of base DEF.

$\therefore$ base $ABC \equiv$ base DEF. (?)

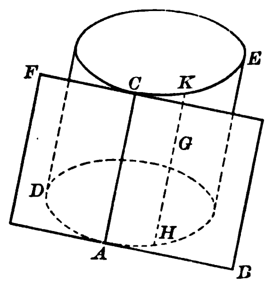
Q. E. D.

Cor. I. *The sections of a cylinder made by parallel planes which cut all the elements are congruent.*

Cor. II. *Any section of a cylinder made by a plane parallel to the bases is congruent to the bases.*

*Proposition 285 Theorem

The plane determined by a tangent to the base of a circular cylinder and the element drawn through the point of contact is tangent to the cylinder.



Hypothesis. AB is tangent to the base of the circular cylinder DE at A, and AC is the element drawn through A. The plane FB is determined by AB and AC.

Conclusion. The plane FB is tangent to the cylinder.

Proof. If possible, suppose that the plane FB is not tangent to the cylinder and meets it at G, a point without AC. Let HK be the element passing through G.

HK lies in the plane FB. (?)

Then AB will meet the perimeter of the base in two points, A and H, which is impossible. (?)

$\therefore$ the plane cannot meet the cylinder in any point without AC, and accordingly is tangent to the cylinder. (?)

Q. E. D.

Cor. *If a plane is tangent to a circular cylinder, its intersection with the plane of either base is tangent to that base.*

Ex. 1138. If a plane is tangent to a circular cylinder, every element of the cylinder except the element of contact is parallel to the plane.

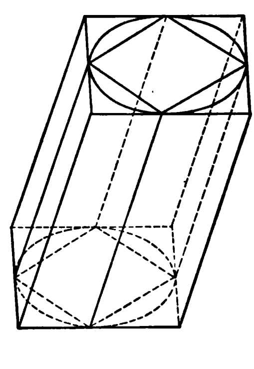
Ex. 1139. If two planes are tangent to a circular cylinder, their line of intersection is parallel to every element of the cylinder.

DEFINITIONS. A prism is said to be *inscribed in a cylinder* when its bases are inscribed in the bases of the cylinder and its lateral edges are elements of the cylinder.

A prism is said to be *circumscribed about a cylinder* when its bases are circumscribed about the bases of the cylinder and its lateral edges are parallel to elements of the cylinder.

Proposition 286 Theorem

If a prism whose bases are regular polygons is inscribed in or circumscribed about a circular cylinder, and the number of its lateral faces is increased indefinitely, the volume of the prism approaches the volume of the cylinder as its limit, and the lateral area of the prism approaches the lateral area of the cylinder as its limit.



Hypothesis. Let V and V' denote the volumes of two prisms whose bases are regular polygons, the former inscribed in a circular cylinder and the latter circumscribed about the same cylinder. Let S and S' denote the lateral areas of the prisms.

Conclusion. As the number of the lateral faces of the prisms is increased indefinitely, the limit of either V or V' is the volume of the cylinder, and the limit of either S or S' is the lateral area of the cylinder.

Proof. The bases of the prisms are regular polygons, the first inscribed in, and the second circumscribed about, the bases of the cylinder. (?)

If the bases of either prism could be made to coincide with the bases of the cylinder, the prism and the cylinder would coincide throughout. Thus the volume and lateral area of the prism would be respectively the same as the volume and lateral area of the cylinder.

Now, as the number of faces of either prism is increased, the bases of the prism approach the bases of the cylinder as limits. (?)

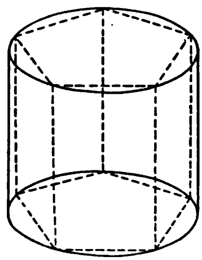
Hence the bases of the prism can be made to come as near as we please to coinciding with the bases of the cylinder, and the difference between either V or V' and the volume of the cylinder can be made so small as to become and remain less than any assigned quantity, however small. That is, the limit of either V or V' is the volume of the cylinder.

Similarly, the limit of either S or S' is the lateral area of the cylinder.

Q. E. D.

Proposition 287 Theorem

The lateral area of a cylinder of revolution is equal to the product of the circumference of the base and the altitude.



Hypothesis. Let S denote the lateral area, c the circumference of the base, and h the altitude of a cylinder of revolution.

Conclusion. $S = c \times h.$

Proof. Inscribe in the cylinder a regular prism; let S' denote the lateral area, and p the perimeter of the base.

The altitude of the prism is h. (?)

Then $S' = p \times h$ (?)

If, now, the number of lateral faces of the prism is increased indefinitely, $S' \doteq S$ and $p \doteq c$. (?)

Then $p \times h \doteq c \times h$. (?)

The numbers which express the values of the variables S' and $p \times h$ are one and the same, whatever the number of lateral faces of the pyramid.

Hence the numbers which express the values of the limits are one and the same, and

$$S = c \times h.$$

Q. E. D.

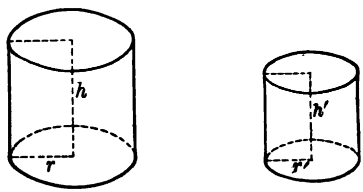
Cor. Let S denote the lateral area, T the total area, r the radius, and h the altitude of a cylinder of revolution.

$$S = 2\pi r h.$$
$$T = 2\pi r h + 2\pi r^2 = 2\pi r(h + r).$$

Remark. The lateral surface of a cylinder of revolution can be exactly covered by a rectangular piece of paper. The dimensions of the sheet are the circumference of the base and the altitude of the cylinder.

Proposition 288 Theorem

The lateral areas, or the total areas, of similar cylinders of revolution are to each other as the squares of their radii, or as the squares of their altitudes.



Hypothesis. Let S and S' denote the lateral areas, T and T' the total areas, r and r' the radii, and h and h' the altitudes of two similar cylinders of revolution.

Conclusion. $$\frac{S}{S'} = \frac{T}{T'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}.$$

Proof. $$\frac{h}{h'} = \frac{r}{r'} = \frac{h+r}{h'+r'}. \qquad (?)$$

$$\frac{S}{S'} = \frac{2\pi rh}{2\pi r'h'} = \frac{r}{r'} \times \frac{h}{h'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}. \qquad (?)$$

$$\frac{T}{T'} = \frac{2\pi r(h+r)}{2\pi r'(h'+r')} = \frac{r}{r'} \times \frac{h+r}{h'+r'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}. \qquad (?)$$

Q.E.D.

Proposition 289 Theorem

The volume of a circular cylinder is equal to the product of its base and its altitude.

Use a method of proof similar to that used in Prop. 287.

Cor. Let V denote the volume, r the radius, and h the altitude of a cylinder of revolution.

$$V = \pi r^2 h.$$

Proposition 290 Theorem

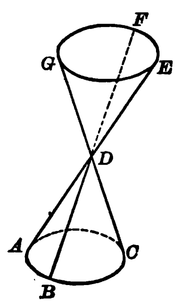
The volumes of similar cylinders of revolution are to each other as the cubes of their radii, or as the cubes of their altitudes.

Use a method of proof similar to that used in Prop. 288.

Ex. 1140. The volume of a cylinder of revolution is equal to one half the product of the lateral area and the radius.

THE CONE

A *conical surface* is a curved surface generated by a straight line which continually moves along a given curve and constantly passes through a given fixed point not coplanar with the curve.



For example, if the straight line AE moves along the curve ABC so that it constantly passes through the point D, the surface generated is a conical surface.

The moving line is called the *generatrix*, and the curve is called the *directrix*. The generatrix in any position is called an *element* of the surface. The fixed point is called the *vertex*.

If the generatrix is of indefinite length, the conical surface consists of two portions on opposite sides of the vertex, which are called the *upper* and *lower nappes*.

A *cone* is a solid bounded by a closed conical surface and a plane. The section of the conical surface formed by the plane is called the *base* of the cone, and the conical surface is called the *lateral surface*. The area of the lateral surface is called the *lateral area* of the cone. An element of the conical surface is called an *element* of the cone, and the vertex of the conical surface is called the *vertex* of the cone.

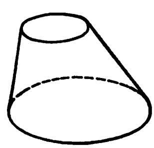
A cone whose base is a circle is called a *circular* cone. The straight line drawn from the vertex to the centre of the base is called the *axis* of the circular cone.

If the axis of a circular cone is perpendicular to the base, it is called a *right* circular cone. If the axis is oblique to the base, it is called an *oblique* circular cone.

The perpendicular distance from the vertex to the base is called the *altitude* of the cone.

A plane is *tangent* to a cone when it passes through an element, but does not meet it again, however far it is produced. The element through which the plane passes is called the *element of contact*.

A *truncated cone* is the part of a cone included between the base and a section cutting all the elements. The base of the cone and the section thus made are respectively the lower base and the upper base of the truncated cone.



A *frustum* of a cone is a truncated cone whose bases lie in parallel planes. The perpendicular distance between the bases of the frustum is called the *altitude*. The portion of the lateral surface of the cone included between the bases of the frustum is called the *lateral surface* of the frustum.

A section of a frustrum of a cone made by a plane parallel to the bases and midway between them is called a *mid-section.*

The following principles follow at once from the preceding definitions:

(i) *The elements of a right circular cone are equal.*

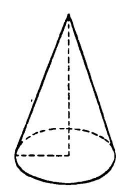
(ii) *The altitude of a right circular cone is the axis of the cone.*

(iii) *A line drawn from the vertex through any point of a conical surface is an element.*

Proposition 291 Theorem

A right circular cone may be generated by the revolution of a right triangle about one of its legs as an axis.

Prove that the solid generated fulfills *all* the requisites of a right circular cone.



Cor. *A frustum of a right circular cone may be generated by the revolution of a trapezoid, one of whose legs is perpendicular to the bases, about that leg as an axis.*

Definitions. On account of the manner in which a right circular cone may be generated, it is also called a *cone of revolution.* The radius of the base is called the *radius* of the cone.

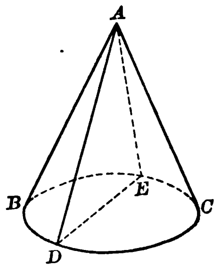
Any element of a cone of revolution is called the *slant height* of the cone. The *slant height* of a frustum of a cone of revolution is the portion of any element of the cone included between the bases.

Similar cones of revolution are cones generated by the revolution of similar right triangles about homologous legs as axes.

A cone of revolution whose section through the axis is an equilateral triangle is called an *equilateral* cone.

Proposition 292 Theorem

Every section of a cone made by a plane passing through its vertex is a triangle.



Hypothesis. ADE is a section of the cone A-BC made by a plane passing through the vertex A.

Conclusion. ADE is a $\triangle$.

Proof. Draw straight lines in the plane ADE from A to D and E.

These lines are elements of the cone. (?)

$\therefore$ these lines are the intersections of the plane ADE and the cone.

DE is a straight line. (?)

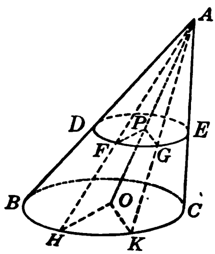
$\therefore ADE$ is a $\triangle$. (?)

Q. E. D.

Cor. *Every section of a cone of revolution made by a plane passing through its vertex is an isosceles triangle.*

Proposition 293 Theorem

Every section of a circular cone made by a plane parallel to the base is a circle, the centre of which is the intersection of the plane with the axis.



Hypothesis. *DFE* is a section of the circular cone *A-BC* made by a plane ∥ to the base, and the axis *AO* intersects this section at *P*.

Conclusion. *DFE* is a ⊙, the centre of which is *P*.

Proof. Let *F* and *G* be any two points in the perimeter of *DFE*.

Through *F* and *G* draw the elements *AH* and *AK*.

Let the plane determined by *AH* and *AO* intersect the base in *OH* and the section in *PF*. Let the plane determined by *AK* and *AO* intersect the base in *OK* and the section in *PG*.

$$PF \text{ is } \parallel \text{ to } OH. \qquad (?)$$

$$\therefore \triangle APF \sim \triangle AOH. \qquad (?)$$

$$\therefore \frac{PF}{OH} = \frac{AP}{AO} \qquad (?)$$

In like manner it can be proved that $\frac{PG}{OK} = \frac{AP}{AO}$.

$$\therefore \frac{PF}{OH} = \frac{PG}{OK}. \qquad (?)$$

But $OH = OK$. (?)

$\therefore PF = PG$. (?)

Since F and G are any two points in the perimeter of DFE, all points in this perimeter are at the same distance from P.

Hence DFE is a ⊙, the center of which is P.

Q. E. D.

COR. I. *If a circular cone is cut by a plane parallel to its base, the elements and the altitude are divided proportionally.*

COR. II. *If a circular cone is cut by a plane parallel to its base, the area of the section is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the cone.*

COR. III. *The areas of two parallel sections of a circular cone are to each other as the squares of their distances from the vertex.*

COR. IV. *If two circular cones have equal altitudes and equal bases, sections made by planes parallel to the bases and at equal distances from the vertices are equal circles.*

COR. V. *If a circular cone is cut by a plane parallel to the base, the cone cut off is similar to the original cone.*

*Proposition 294 Theorem

The plane determined by a tangent to the base of a circular cone and the element drawn through the point of contact is tangent to the cone.

Use a method of proof similar to that used in Prop. 285.

COR. I. *If a plane is tangent to a circular cone, its intersection with the plane of the base is tangent to the base.*

COR. II. *Every plane tangent to a cone passes through the vertex of the cone.*

Ex. 1141. If two planes tangent to a cone of revolution pass through two diametrically opposite elements, the intersection of the planes is a line passing through the vertex parallel to the base of the cone.

DEFINITIONS. A pyramid is said to be *inscribed in a cone* when its base is inscribed in the base of the cone and its lateral edges are elements of the cone.

A pyramid is said to be *circumscribed about a cone* when its base is circumscribed about the base of the cone and its vertex coincides with the vertex of the cone.

If a pyramid is inscribed in or circumscribed about a cone, the plane which cuts off a frustum of the cone also cuts off a frustum of the pyramid, which is said to be *inscribed in* or *circumscribed about* the frustum of the cone.

Proposition 295 Theorem

If a pyramid whose base is a regular polygon is inscribed in or circumscribed about a circular cone, and the number of its lateral faces is increased indefinitely, the volume of the pyramid approaches the volume of the cone as its limit, and the lateral area of the pyramid approaches the lateral area of the cone as its limit.

The method of proof is the same as that of Prop. 286.

COR. *If a frustum of a pyramid whose bases are regular polygons is inscribed in or circumscribed about a frustum of a cone, and the number of its lateral faces is increased indefinitely, the volume of the frustum of the pyramid approaches the volume of the frustum of the cone as its limit, and the lateral area of the frustum of the pyramid approaches the lateral area of the frustum of the cone as its limit.*

Proposition 296 Theorem

The slant height of a regular pyramid circumscribed about a cone of revolution is equal to the slant height of the cone.

Proposition 297 Theorem

The lateral area of a cone of revolution is equal to one half the product of the circumference of its base and its slant height.

Let a regular pyramid be circumscribed about the cone. Use a method of proof similar to that used in Prop. 287.

COR. Let S denote the lateral area, T the total area, r the radius of the base, and l the slant height of a cone of revolution.

$$S = \tfrac{1}{2}(2\pi r \times l) = \pi rl.$$
$$T = \pi rl + \pi r^2 = \pi r(l + r).$$

REMARK. The lateral surface of a cone of revolution can be exactly covered by a piece of paper in the shape of a sector of a circle. The arc of the sector is equal to the circumference of the base of the cone and the radius of the sector is equal to an element of the cone.

Proposition 298 Theorem

The lateral areas, or the total areas, of similar cones of revolution are to each other as the squares of their radii, or as the squares of their altitudes.

Use a method of proof similar to that used in Prop. 288.

Ex. 1142. If the slant height of a cone of revolution is equal to the diameter of the base, the lateral area is double the area of the base.

Ex. 1143. The lateral area of a cone of revolution is equal to the area of a circle whose radius is the mean proportional between the slant height and the radius of the base.

Proposition 299 Theorem

The volume of a circular cone is equal to one third the product of its base and altitude.

Let a pyramid whose base is a regular polygon be inscribed in the cone. Use a method of proof similar to that used in Prop. 287.

Cor. Let V denote the volume, r the radius of the base, and h the altitude of a cone of revolution.

$$V = \frac{1}{3}\pi r^2 h.$$

Proposition 300 Theorem

The volumes of similar cones of revolution are to each other as the cubes of their radii, or as the cubes of their altitudes.

Use a method similar to that used in Prop 288.

Ex. 1144. The volume of a cone of revolution is equal to one third the product of the lateral area and the distance of any element from the center of the base.

Ex. 1145. The volume of a cone of revolution is equal to the product of the area of its generating triangle and the circumference of the circle whose radius is the distance from the intersection of the medians of the triangle to the axis.

Proposition 301 Theorem

The lateral area of a frustum of a cone of revolution is equal to one half the product of the sum of the circumferences of its bases and its slant height.

Let a frustum of a regular pyramid be circumscribed about the frustum of the cone. Use a method of proof similar to that used in Prop. 287.

Cor. I. *The lateral area of a frustum of a cone of revolution is equal to the product of the circumference of its mid-section and its slant height.*

Cor. II. Let S denote the lateral area, r and r' the radii of the bases, and l the slant height of a frustum of a cone of revolution.

$$S = \tfrac{1}{2}(2\pi r + 2\pi r') \times l = \pi(r + r')l.$$

Proposition 302 Theorem

A frustum of a circular cone is equivalent to the sum of three cones whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.

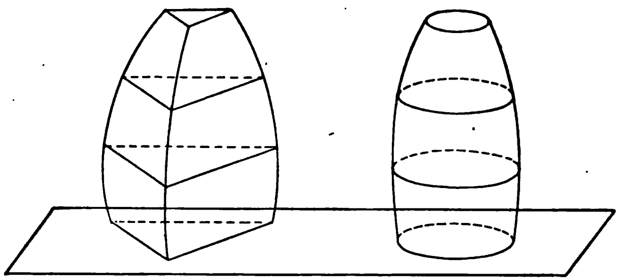
Let a frustum of a regular pyramid be inscribed in the frustum of the cone.

Cor. Let V denote the volume, r and r' the radii of the bases, and h the altitude of a frustum of a circular cone.

$$\begin{aligned} V &= \tfrac{1}{3}h(\pi r^2 + \pi r'^2 + \sqrt{\pi r^2 \times \pi r'^2}) \\ &= \tfrac{1}{3}\pi h(r^2 + r'^2 + rr'). \end{aligned}$$

Proposition 303 Theorem

If two solids are included between parallel planes and if sections of the solids made by any plane parallel to the given planes are equivalent, the two solids are equivalent.



Hypothesis. Let V and V' denote the volumes of two solids which lie between ∥ planes, sections of these solids made by any plane ∥ to the given planes being equivalent.

Conclusion. $V = V'$.

Proof. Let h denote the common altitude of the two solids, and divide h into any number of equal parts.

Through the points of division pass planes ∥ to the given ∥ planes.

Using these sections as lower bases construct a series of solids having altitudes equal to the divisions of h.

Each solid thus constructed is a prism, a cylinder, or a combination of the two. In each case the volume is the product of the base and altitude, and the corresponding small solids are equivalent. (?)

Let W and W' denote the sums of the small solids constructed in the two figures.

$$\text{Then } W = W' \qquad (?)$$

If, now, the number of parts into which h is divided is increased indefinitely, the differences between W and V and between W' and V' can be made so small as to become and remain less than any assigned quantity, however small.

$$\therefore W \doteq V \text{ and } W' \doteq V'. \qquad (?)$$

The numbers which express the values of the variables W and W' are one and the same, whatever the number of parts into which h is divided.

Hence the numbers which express the values of the limits are one and the same, and

$$V = V'.$$

Q. E. D.

Remark. Proposition 303 is commonly known as *Cavalieri's Theorem.* By means of this theorem it can be shown that the Prismatoidal Formula applies to certain solids having curved sides. If a circular cone is compared with a pyramid having an equivalent base and an equal altitude, reference to Prop. 262, Prop. 293, Cor. II, and Ax. 4 shows that the volume of the cone is equal to the volume of the pyramid. Accordingly, the Prismatoidal Formula can be used to determine the volume of a cone or of a frustum of a cone. Likewise, by comparing with a prism, it can be shown that the volume of a cylinder can be determined by this formula.

Ex. 1146. By making the proper substitution in the prismatoidal formula, derive the formula for a cylinder, as given on page 376.

Ex. 1147. By making the proper substitution in the prismatoidal formula, derive the formula for a cone, as given on page 384.

Ex. 1148. By making the proper substitution in the prismatoidal formula, derive the formula for a frustum of a cone, as given on page 385.

EXERCISES

Ex. 1149. Through a given point in a cylindrical surface one straight line can be drawn in that surface, and only one.

Ex. 1150. The lateral area of a cylinder of revolution is equal to the area of a circle whose radius is a mean proportional between the altitude of the cylinder and the diameter of its base.

Ex. 1151. If the altitude of a cylinder of revolution is equal to the diameter of the base, the volume is equal to one third the product of the total area and the radius.

Ex. 1152. If two cylinders of revolution have equal lateral areas, their volumes are in the same ratio as their radii.

Ex. 1153. If two cylinders of revolution have equal volumes, their total areas and their altitudes are reciprocally proportional.

Ex. 1154. Every section of a frustum of a cone made by a plane passing through an element is a trapezoid.

Ex. 1155. The radii of the upper and lower bases of a frustum of a circular cone have the same ratio as the distances of the bases from the vertex of the cone.

Ex. 1156. Find the locus of a point at a given distance from a given straight line.

Ex. 1157. Find the locus of a point at a given distance from a given cylindrical surface.

Ex. 1158. Find the locus of a line which makes a given angle with a given line at a given point.

Ex. 1159. Find the locus of a line which makes a given angle with a given plane at a given point.

Ex. 1160. A line moves so that it always passes through a fixed point without a fixed cylindrical surface and is tangent to the surface. Find the locus (i) of the point of contact; (ii) of the moving line.

Ex. 1161. Find the locus of a point whose distance from a fixed line is equal to its distance from a fixed plane perpendicular to that line.

Ex. 1162. From a point A outside a plane a line is drawn meeting the plane at P and making an angle of 45° with the plane. Find the locus (i) of the point P; (ii) of the line AP.

PROBLEMS OF COMPUTATION

[Use for π the approximate value $3\frac{1}{7}$]

1. Find the total area and the volume of a cylinder of revolution whose radius is 7 in. and whose altitude is 12 in.

2. The altitude of a cylinder of revolution is 10 in.. and the circumference of its base is 26.4 in.; find its volume.

3. Find the volume of a cylinder of revolution whose total area is 484 sq. in., and whose altitude is 4 in.

4. A cylindrical tin pail 7 in. in diameter contains water to the depth of 4 in. An egg is then immersed in the water, and the level of the latter rises to 4.22 in. above the bottom of the pail. Find the volume of the egg.

5. The altitude and the diameter of a cylinder of revolution are equal, and the number of square feet in the total area is the same as the number of cubic feet in its volume. Find its altitude.

6. Find the length of wire $\frac{1}{10}$ of an inch in diameter that can be made from a cubic foot of copper.

7. It is desired to cut off a piece of lead pipe 2 in. in outside diameter and $\frac{1}{4}$ in. thick so that it will melt into a cube whose edge is 6 in. How long a piece will be required?

8. Find the number of cubic yards of earth removed in digging a tunnel 175 yd. long if the cross section is a semicircle with a radius of 14 ft.

9. A cross section of a tunnel is a rectangle 8 ft. by 12 ft., surmounted by a semicircle whose diameter is the smaller dimension of the rectangle. If the tunnel is three fourths of a mile long, how many cubic yards of earth were removed?

10. A rectangle, whose dimensions are a and b, revolves successively about two adjacent sides as axes. Find the ratio of the volumes of the two solids generated.

11. If a square is rotated completely about a line in its plane not crossing it, but parallel to one of its sides, find the volume of the solid thus generated.

12. The volume of a cylinder of revolution is V, and its altitude is h; find its total area.

13. The total area of a cylinder of revolution is T, and its lateral area is S; find its volume.

14. If one edge of a cube is a, find the radius of an equivalent cylinder whose altitude is h.

15. A grocer cuts off one fourth of the circumference of a twelve pound cylindrical cheese at one straight cut. Find the weight of the piece cut off.

16. A railroad oil tank has the shape of a right circular cylinder with its axis horizontal. The internal diameter of the tank is 6 ft. and its length is 25 ft. How many gallons will it hold if filled to the depth of $4\frac{1}{2}$ ft.?

17. Find the total area and the volume of a cone of revolution whose altitude is 12 ft., and the diameter of whose base is 10 ft.

18. The volume of a cone is 352 cu. in. and its radius is 4 in.; find its altitude.

19. Find the total area of a cone of revolution whose volume is 308 cu. in., and whose altitude is 6 in.

20. A conical tent, whose slant height is 14 ft., contains 176 sq. ft. of canvas. Find the area of the ground covered by the tent.

21. The altitude of a cone of revolution is equal to the diameter of the base; find the ratio of the area of the base to the lateral area.

22. The lateral area of a cone of revolution is twice the area of the base; if the radius is r, find the volume.

23. The lateral area of a cone of revolution is S, and its radius is r; find its volume.

24. The lateral area of a right circular cone is equal to three times the area of the base. If the radius of the base is r, find the altitude and the volume of the cone.

25. The altitude of a cone, the diameter of its base, and the edge of a given cube are equal. Find the ratio of the volume of the cone to the volume of the cube.

26. The altitude of a cone of revolution is 12 in. and the radius of its base is 5 in. Find the radius of the sector of paper which, when rolled up, will just cover the curved surface of the cone; also find the angle of the sector.

27. A hollow cone of revolution whose altitude is equal to three fourths of the slant height is cut open in a straight line drawn from the vertex to a point in the base. Find the vertical angle of the unrolled surface.

28. A can buoy is made by cutting a circular sheet of iron 2 yd. in diameter into two semicircles, bending each half into a conical surface, and fastening these surfaces together base to base. Find the volume of the buoy.

29. A lead pencil $\frac{1}{4}$ of an inch in diameter is sharpened so that its end has the form of a right circular cone whose altitude is $\frac{3}{8}$ of an inch. If the pencil is resharpened in the same way, beginning at a point $\frac{1}{8}$ of an inch further up the pencil, how much material is removed?

30. The legs of a right triangle are a and b. Find the ratio of the volumes of the solids of revolution generated by revolving the triangle about these legs as axes.

31. An equilateral triangle, whose side is 7 in., revolves about its altitude as an axis. Find the total surface of the solid generated.

32. A right triangle whose altitude is 4 in. and whose area is 6 sq. in. is revolved about its shortest side as an axis. Find the volume of the solid thus generated.

33. What is the volume of the solid of revolution generated by revolving a right triangle about its hypotenuse as an axis, the length of the hypotenuse being 13 in., and the lengths of the two legs being 5 in. and 12 in.?

34. A double cone is generated by the revolution of an isosceles triangle about its base. The volume of the solid is 22 cu. ft., and the length of the base of the triangle is 4 ft. 8 in. Find the altitude of the triangle.

35. A rectangle is divided into two triangles by a diagonal. Find the ratio of the volumes generated by these two triangles as the rectangle revolves about the longer side as an axis.

36. A square, whose area is S, revolves about its diagonal as an axis. Find the volume of the solid generated.

37. The base of an isosceles triangle is 6 in., and the legs are each 5 in. A line is drawn parallel to the base through the opposite vertex, and the triangle is revolved about this line as an axis. Find the volume of the solid thus generated.

38. From the vertices of the triangle ABC perpendiculars AA', BB', and CC' are drawn to a straight line XY in the same plane. $AA' = 2$ in., $BB' = 3$ in., $CC' = 1$ in., $A'B' = 2$ in., $B'C' = 1$ in., and $C'A' = 3$ in. Find the volume generated by revolving ABC about XY as an axis.

39. The diameter of the base of a right circular cone is 10.24 ft. and its altitude is 18.3 ft. Find the altitude of a right circular cylinder of equivalent volume, the diameter of whose base is 14.38 ft.

40. A projectile has the shape of a cylinder of revolution surmounted by a conical cap. Its total length is 24 in., and the cylindrical part is three times as long as the conical end. The greatest diameter is 10 in. Find the volume and the total surface of the projectile.

41. From a cylinder of revolution whose altitude is 12 in. and whose diameter is 10 in. is cut out a cone. The base of the cone coincides with one base of the cylinder, and its vertex is at the center of the other base of the cylinder. Find the volume and the total surface of the solid which remains.

42. The radius of the base of a cone of revolution is 5 in. and its altitude is 10 in. A cylinder of revolution is inscribed in the cone, its base lying in the base of the cone. The altitude of the cylinder is 4 in. Find the ratio of the volumes of the two solids; also find the ratio of their lateral surfaces.

43. If an equilateral cylinder and an equilateral cone have the same total surface, find the ratio of their volumes.

44. A right circular cylinder and a right circular cone have the same altitude h and the same base. If the difference between the areas of their curved surfaces is equal to the area of their common base, find the radius of the base.

45. A cone of revolution whose height is 10 in., and whose radius is 5 in., has a round hole 1 in. in diameter bored through it, the axis of the hole coinciding with the axis of the cone. What is the volume of the wood removed?

46. Find the total area and the volume of a frustum of a cone of revolution whose altitude is 3 ft., and the radii of whose bases are 1 ft. and 2 ft.

47. Find the total area and the volume of a frustum of a cone of revolution whose slant height is 6.25 in., and the radii of whose bases are 4.8 in. and 5.6 in.

48. An open cistern has the form of a frustum of a cone of revolution; the depth of the cistern is 7 ft., and the top and bottom diameters are 3 ft. and 6 ft. respectively. Find the cost of painting the inside of this cistern at $1\frac{1}{2}$ cents per square foot. How many gallons will the cistern hold?

49. How much tin will it take to make a milk pail, if the diameter of the base is 1 ft., the diameter of the top is 16 in., and the slant height is 14 in.? How much milk will the pail hold?

50. A tank in the form of a frustum of a cone is 7 ft. deep. Its diameter at the top is 6 ft.; at the bottom 6 ft. 8 in. How many gallons will it hold?

51. What must be the depth of a pail that measures 18 in. across the top and 10 in. across the bottom in order that it may hold 5280 cu. in.?

52. The altitude of the frustum of a right circular cone is 10 ft., and the areas of the bases are 289 sq. ft. and 961 sq. ft. A plane is passed parallel to the bases cutting from the frustum a section whose area is one half the sum of the two bases. Find the distance from the lower base to this plane.

53. It is desired to cut out from a flat piece of cardboard a piece which can be bent into the form of the curved surface of a right circular cone whose elements make with the axis angles of 30°, and whose base has a radius of 5 in. Show how this can be done.

How should this piece of cardboard be further cut, while still flat, so that when bent it takes the form of a lamp shade obtained by cutting off the apex from the cone described above by a plane parallel to the base and 5 in. above it?

54. The altitude of a cylinder of revolution is h, and the radius of the base is r. Through a tangent line to the lower base a plane is passed making an angle of 30° with the plane of this base. What part of the volume of the cylinder lies below this plane?

55. The altitude of a right cone is 10 in.; how far from the vertex of the cone must two planes be passed parallel to the base in order to divide the lateral surface into three equivalent parts?

56. If the slant height of a cone of revolution is 18 in., find the lengths of the two parts into which the slant height is divided by a plane parallel to the base which divides the curved surface in the ratio 3 : 5.

57. At what distance from the vertex must a right cone 6 in. high be cut by a plane parallel to the base in order to divide it into two equivalent volumes?

58. A conical vessel full of water is 8 in. deep and measures 6 in. across the top. Water is drawn off until the level has sunk 2 in. Find the amount of water remaining.

59. A hollow glass cone holds one quart. Its inside slant height is 10 in. How shall it be graduated to measure a pint?

60. A right circular cone stands in water which comes half way up to its vertex. What part of the volume and of the lateral surface is submerged?

61. A right circular cone, whose height is 16 in., and whose base has the area 25 sq. in., is cut by a plane parallel to the base so that the section has the area 9 sq. in. What is the altitude of the remaining frustum?

62. The altitude of a cone of revolution is 20 in., and its radius is 10 in. What must be the distance from the base of a plane parallel to the base in order that the volume of the frustum thus formed may be 80 cu. in.?

63. The diameter and depth of a conical cup are equal. If filled with water to half its depth, what is the ratio of its contents to its full capacity? What is the ratio of the wet surface to the dry surface?

64. A plane parallel to the base of a cone of revolution cuts off a frustum whose volume is $203\frac{1}{3}$ cu. in., and the radii of whose bases are 4 in. and 3 in. Find the volume of the cone.

65. The radii of the bases of a frustum of a circular cone are 5 in. and 2 in., and the altitude of the frustum is 3 in. Find the volume of the completed cone.

66. A certain plane parallel to the base of a cone of revolution divides the lateral area of the cone into two equivalent parts. Find the distance from the vertex to this plane in terms of the altitude h. If the section cut by this plane from the cone is the base of a cylinder of altitude h, find the ratio of the volume of cylinder to the volume of the cone.

67. A cylindrical vessel is 12 in. in diameter and 8 in. deep. What are the diameter and depth of a similar vessel which will hold only one sixty-fourth as much?

68. The volume of a cone is V; what is the volume of a cone whose surface is n times as large?

69. The total areas of two similar cylinders of revolution are 75 sq. in. and 192 sq. in. If the volume of the first cylinder is 250 cu. in., what is the volume of the second?

Book IX

THE SPHERE

A *sphere* is a closed surface, all points of which are equally distant from a point within called the *centre*. Accordingly, *the locus of a point in space at a given distance from a given point is a sphere.*

Note. The word *sphere* was formerly used in elementary mathematics to denote the solid bounded by the curved surface. In recent years the surface has been called a sphere in accordance with the usage of higher mathematics.

A straight line drawn from the centre to any point of the sphere is called a *radius*. A straight line passing through the centre and having its extremities in the *sphere* is called a *diameter*.

The following principles follow at once from the preceding definitions:

(i) *All radii of a sphere are equal.*

(ii) *A diameter of a sphere is equal to the sum of two radii.*

(iii) *All diameters of a sphere are equal.*

(iv) *A point is within a sphere, on a sphere, or without a sphere, according as its distance from the centre is less than, equal to, or greater than a radius.*

(v) *Two spheres are equal when they have equal radii.*

(vi) *Radii of equal spheres are equal.*

(vii) *A sphere may be generated by the revolution of a semicircle about its diameter as an axis.*

Note. Equal spheres are at the same time congruent; hence in speaking of congruent spheres the simpler word *equal* is generally used.

A straight line or a plane which has one point in common with a sphere, but does not meet it again, however far it is produced, is said to be *tangent* to the sphere, or to *touch* it. The common point is called the *point of contact*, or *point of tangency.*

Two spheres are said to be *tangent* to each other, or to *touch* each other, when they have one point in common, and only one.

A polyhedron is said to be *inscribed in a sphere* when all its vertices lie on the sphere. The sphere is said to be *circumscribed about* the polyhedron.

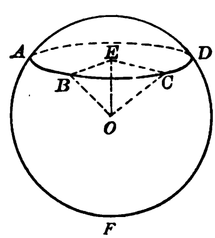
A polyhedron is said to be *circumscribed about a sphere* when all its faces are tangent to the sphere. The sphere is said to be *inscribed in* the polyhedron.

A cylinder or cone is said to be *circumscribed about a sphere* when its bases (or base) and all its elements are tangent to the sphere. The sphere is said to be *inscribed in* the cylinder or cone.

Spheres which have the same centre are said to be *concentric.*

Proposition 304 Theorem

Every section of a sphere made by a plane is a circle.



Hypothesis. In the sphere *ADF*, of which *O* is the centre, *ACD* is a section made by a plane.

Conclusion. *ACD* is a ⊙.

Proof. Let *B* and *C* be any two points in the perimeter of *ACD*.

Draw *OE* ⊥ to the plane *ACD*. Draw *OB*, *OC*, *EB*, and *EC*.

[To be completed by the student.]

Definitions. Any circular section of a sphere made by a plane is called a *circle of the sphere.* If the plane of a section passes through the centre of the sphere, the section is a circle with a radius equal to the radius of the sphere; such a circle is called a *great circle.* A section made by a plane which does not pass through the centre of the sphere is called a *small circle.*

The diameter of a sphere which is perpendicular to the plane of a circle is called the *axis* of the circle, and its extremities are called the *poles* of the circle.

A cylinder is said to be *inscribed in* a sphere when its bases are circles of the sphere. The sphere is said to be *circumscribed about* the cylinder.

A cone is said to be *inscribed in* a sphere when its base is a circle of the sphere and its vertex is a point on the sphere. The sphere is said to be *circumscribed about* the cone.

Note. In Spherical Geometry, the word *quadrant* is used to denote a quadrant of a great circle.

Proposition 305 Theorem

The axis of any circle of a sphere passes through the centre of the circle.

Proposition 306 Theorem

Any great circle bisects the sphere.

Use the method of superposition.

DEFINITION. Either half of a sphere is called a *hemisphere.*

Proposition 307 Theorem

Any two great circles on the same sphere bisect each other.

COR. I. *All great circles of a sphere are equal.*

COR. II. *The planes of two great circles of a sphere intersect in a diameter.*

Proposition 308 Theorem

Through any three points of a sphere one circle can be drawn, and only one.

Proposition 309 Theorem

Through any two points of a sphere, not at the extremities of a diameter, one great circle can be drawn, and only one.

REMARK. If the two given points are at the extremities of a diameter, an indefinite number of great circles can be drawn through them.

DEFINITION. The *spherical distance* between two points on a sphere is the length of the minor arc of the great circle joining the two points. It will be proved later that this line is the shortest line that can be drawn between the two given points.

Ex. 1163. If the planes of two circles of a sphere are equally distant from the center of the sphere, the circles are equal.

Ex. 1164. State and prove the converse of Ex. 1163.

Ex. 1165. If the planes of two circles of a sphere are unequally distant from the center of the sphere, the circle at the less distance is the greater.

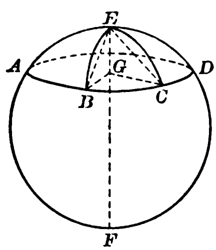
Ex. 1166. State and prove the converse of Ex. 1165.

Ex. 1167. If two circles of a sphere lie in parallel planes, they have the same axis and the same poles.

Ex. 1168. If one great circle of a sphere passes through the pole of another great circle, their planes are perpendicular to each other.

Proposition 310 Theorem

All the points in a circle of a sphere are equally distant from either of the poles of the circle.



Hypothesis. On the sphere ADF, E is a pole of the $\odot ABD$, and B and C are any two points in the circle.

Conclusion. B and C are equally distant from E.

Proof. Let G be the centre of the $\odot ABD$.

Draw GB, GC, EB, and EC.

[To be completed by the student.]

Cor. *All arcs of great circles drawn from a pole of a circle to points in the circle are equal.*

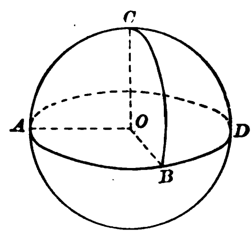
Definition. The arc of a great circle drawn from the nearest pole of a circle of a sphere to any point in the circle is called the *polar distance* of the circle.

Proposition 311 Theorem

The polar distance of a great circle is a quadrant.

Proposition 312 Theorem

If a point on a sphere is at a quadrant's distance from each of two other points on the sphere, not the extremities of a diameter, it is a pole of the great circle passing through these points.



Hypothesis. On the sphere ACD, of which O is the centre, A and B are two points not the extremities of a diameter, and C is at a quadrant's distance from A and B.

Conclusion. C is a pole of the great ⊙ passing through A and B.

Proof. Draw the radii OA, OB, and OC.

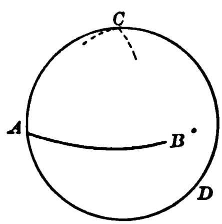
[To be completed by the student.]

Cor. *A chord of a quadrant is equal to the hypotenuse of a right triangle each of whose legs is equal to a radius.*

REMARK. An arc of a circle can be drawn upon a sphere as easily as upon a plane surface. To do this it is necessary to know the position of the pole and the length of the chord of the polar distance. The construction is performed most easily by the use of a pair of compasses with curved branches. The opening of the compasses (the distance between their points) is made equal to the chord of the polar distance. Then placing one point on the pole, the other point describes the circle.

* Proposition 313 Theorem

To describe an arc of a great circle through any two points on a sphere.



To describe an arc of a great ⊙ through A and B, two points on the sphere ACD.

Construction. With A and B as poles and with an opening of the compasses equal to a chord of a quadrant, describe arcs intersecting at C.

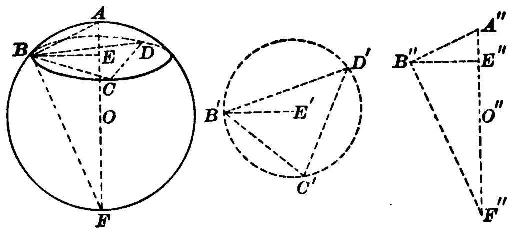
With C as a centre and with the same opening of the compasses, describe the arc AB.

Then AB is the required arc.

[The proof is left to the student.]

* Proposition 314 Theorem

To find the radius of a given sphere.



To find the radius of the sphere ABF.

Construction. With A, any point on the sphere, as a pole and with any opening of the compasses, describe the ⊙ BCD on the sphere. Then the straight line AB is a known line.

Take any three points B, C, and D in this ⊙, and measure with the compasses the straight lines BC, BD, and CD.

On a plane surface construct the △ $B'C'D'$ with sides equal to BC, BD, and CD respectively. Circumscribe a ⊙ about this △, and find its centre E'. Draw the radius $B'E'$.

Construct the rt. △ $A''B''E''$, having the hypotenuse $A''B'' = AB$ and the leg $B''E'' = B'E'$. Draw $B''F'' \perp$ to $A''B''$, meeting $A''E''$ produced at F''. Bisect $A''F''$ at O''.

Then $A''O''$ is equal to the radius of the given sphere.

Proof. In the sphere suppose AF, the axis of the ⊙ BCD, drawn cutting the plane of BCD at E. Also suppose BE and BF drawn.

Then E is the centre of the ⊙ BCD. (?)

$$\triangle B'C'D' \equiv \triangle BCD. \qquad (?)$$

Then the ⓢ circumscribed about the ⊿s $B'C'D'$ and BCD are equal, and $B'E' = BE$. (?)

AE is $\perp$ to BE. (?)

Then $\triangle A''B''E'' \equiv \triangle ABE$. (?)

$\angle ABF$ is a rt. $\angle$. (?)

Then $\triangle A''B''F'' \equiv \triangle ABF$. (?)

$\therefore A''F'' = AF$. (?)

That is, $A''F''$ is equal to the diameter of the sphere, and $A''O''$, which is equal to $2A''F''$, is equal to the radius.

Q. E. D.

REMARK. This method can be used on a material sphere, when only measurements on its surface are possible.

Proposition 315 Theorem

A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.

Use a method of proof similar to that used in Prop. 116.

COR. *A line perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*

Proposition 316 Theorem

A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.

COR. I. *A line tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

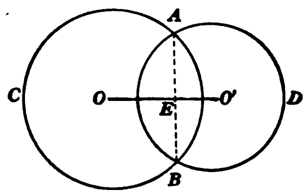
COR. II. *Two lines tangent to a sphere at the same point determine the tangent plane at that point.*

COR. III. *Any line in a tangent plane passing through the point of contact is tangent to the sphere at that point.*

Ex. 1169. All the tangents drawn to a sphere from the same point without the sphere are equal, and touch the sphere in a circle of the sphere.

Proposition 317 Theorem

The intersection of two spheres is a circle whose axis is the line of centres of the spheres.



Hypothesis. O and O' are the centres of two spheres.

Conclusion. The intersection of these spheres is a ⊙ whose axis is the line OO'.

Proof. Let a plane passing through O and O' intersect the spheres in the great ⓢ ABC and ABD. Let A and B be the points of intersection of these ⓢ.

Draw AB intersecting OO' at E.

Then OO' is the ⊥ bisector of AB, and $AE = 2AB$. (?)

Let the plane of these two great ⓢ revolve about OO' as an axis, thus generating two spheres.

The point A will generate the line of intersection of the spheres.

During the revolution AE is always ⊥ to OO'; hence it generates a plane ⊥ to OO'. (?)

Also AE remains constant in length; hence the intersection is a ⊙ with centre at E. (?)

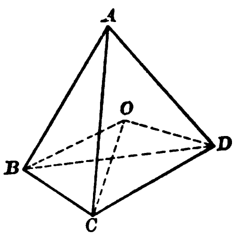
Hence the intersection is a ⊙ whose axis is the line OO'.

Q. E. D.

Ex. 1170. If the distance between the centers of two spheres is equal to the sum of their radii, the spheres are tangent to each other.

Proposition 318 Theorem

A sphere can be inscribed in any tetrahedron.



Hypothesis. *A-BCD* is a tetrahedron.

Conclusion. A sphere can be inscribed in *A-BCD*.

Proof. Let the dihedral ∡ *BC*, *CD*, and *BD* be bisected by the planes *OBC*, *OCD*, and *OBD* respectively. Let *O* be the point of intersection of these three planes.

Since *O* lies in the plane *OBC*, it is equidistant from the planes *ABC* and *BCD*. (?)

Likewise, *O* is equidistant from the planes *ACD* and *BCD*, and from the planes *ABD* and *BCD*.

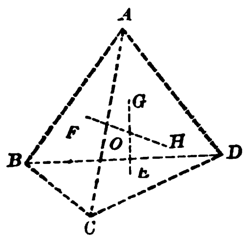
∴ *O* is equidistant from the four faces of the tetrahedron, and a sphere, whose centre is *O* and whose radius is equal to the distance from *O* to either face, will be tangent to each face, and hence will be inscribed in the tetrahedron. (?)

Q. E. D.

Cor. *The six planes which bisect the six dihedral angles of a tetrahedron are concurrent.*

Proposition 319 Theorem

Through any four points not in the same plane, one sphere can be drawn, and only one.



Hypothesis. A, B, C, and D are four points not in the same plane.

Conclusion One sphere can be drawn through A, B, C, and D, and only one.

Proof. Draw AB, AC, AD, BC, BD, and CD, thus forming the tetrahedron A–BCD.

Let E and F be the centres of the ⓢ circumscribed about the ⚠ BCD and ABC respectively.

At E erect $EG \perp$ to the plane BCD. At F erect $FH \perp$ to the plane ABC.

All points in EG are equidistant from B, C, and D, and all points in FH are equidistant from A, B, and C. (?)

∴ all points in both EG and FH are equidistant from B and C.

∴ EG and FH lie in the plane which is the $\perp$ bisector of BC. (?)

EG and FH are not ||. (?)

∴ EG and FH intersect. Let O be the point of intersection.

O is equidistant from A, B, C, and D. (?)

∴ a sphere whose centre is O and whose radius is equal to the distance from O to A will pass through A, B, C, and D.

Moreover, since EG and FH can intersect in only one point, O is the only point equidistant from A, B, C, and D, and only one sphere can be drawn through these points.

Q. E. D.

COR. I. *A sphere can be circumscribed about any tetrahedron.*

COR. II. *The four perpendiculars to the faces of a tetrahedron erected at the centres of their circumscribed circles are concurrent.*

COR. III. *The six planes perpendicular to the edges of a tetrahedron at their mid-points are concurrent.*

Ex. 1171. Find the locus of the center of a sphere which passes through three fixed points.

Ex. 1172. Find the locus of the center of a sphere which is tangent to three fixed planes.

SPHERICAL ANGLES

The *angle of two curves* passing through the same point is the angle formed by two tangents to the curves at that point.

A *spherical angle* is the angle formed by two intersecting arcs of great circles on the surface of a sphere.

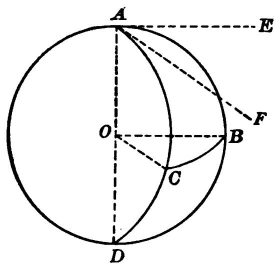
Proposition 320 Theorem

A spherical angle has the same measure as the dihedral angle formed by the planes of the two arcs forming the angle.

Consult Prop. 117 and Prop. 230.

Proposition 321 Theorem

A spherical angle is measured by the arc of a great circle described with its vertex as a pole and included between its sides, produced if necessary.



Hypothesis. On the sphere ABD, of which O is the centre, ABD and ACD are great ⊙ arcs forming the spherical $\angle BAC$, and BC is great ⊙ arc described with A as a pole and included between ABD and ACD.

Conclusion. $\angle BAC \mathrel{\underline{\propto}} \widehat{BC}$.

Proof. Draw AE and AF tangent to the arcs ABD and ACD respectively. Draw the diameter AD and the radii OB and OC.

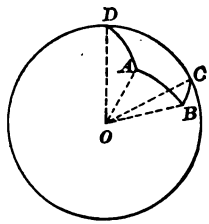
$\measuredangle BAC$ and EAF are identical.	(?)
AE and AF are $\perp$ to AD.	(?)
AD is $\perp$ to the plane of $\widehat{BC}$.	(?)
$\therefore AD$ is $\perp$ to OB and OC.	(?)
$\therefore AE$ is $\parallel$ to OB, and AF is $\parallel$ to OC.	(?)
$\therefore \angle EAF = \angle BOC$.	(?)
But $\angle BOC \mathrel{\underline{\propto}} \widehat{BC}$.	(?)
$\therefore \angle BAC \mathrel{\underline{\propto}} \widehat{BC}$.	(?)

Q. E. D.

COR. *Any arc of a great circle drawn through the pole of a given great circle is perpendicular to the given great circle.*

SPHERICAL POLYGONS AND PYRAMIDS

A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of great circles; as $ABCD$. The bounding arcs are called the *sides* of the polygon, the points of intersection of the arcs are called the *vertices* of the polygon, and the spherical angles formed by the arcs are called the *angles* of the polygon.



An arc of a great circle joining any two vertices of a polygon which are not consecutive is called a *diagonal*.

NOTE. It is customary to express the values of the sides of a spherical polygon in degrees, minutes, and seconds.

The planes of the sides of a spherical polygon form a polyhedral angle whose vertex is the centre of the sphere and whose edges are the radii drawn from the centre of the sphere to the vertices of the polygon. The polyhedral angle is said to *correspond* to the spherical polygon. For example, the polyhedral angle $O\text{-}ABCD$ corresponds to the spherical polygon $ABCD$.

The face angles of the polyhedral angle are measured by the sides of the spherical polygon, and the dihedral angles of the polyhedral angle have the same measures as the angles of the spherical polygon. Accordingly, from any property of polyhedral angles an analogous property of spherical polygons can be deduced; and conversely, from any property of spherical polygons an analogous property of polyhedral angles can be deduced.

A spherical polygon is *convex* when its corresponding polyhedral angle is convex.

In a polyhedral angle each face angle is assumed to be less than two right angles; accordingly, each side of a spherical polygon is assumed to be less than a semicircle.

A *spherical triangle* is a spherical polygon of three sides. The terms *scalene, isosceles, equilateral, equiangular,* and *right* are used with the same meanings as in plane triangles.

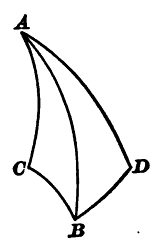
A *spherical pyramid* is a solid bounded by a spherical polygon and the planes of its sides. The centre of the sphere is the *vertex* of the pyramid, and the spherical polygon is its *base*. The lateral edges of the pyramid are radii of the sphere.

Arcs of great circles on a sphere can be superposed and made to coincide in the same way that straight lines are superposed and made to coincide.

If two polygons have their corresponding parts equal and arranged in the same order, they will coincide when one is placed upon the other; hence the polygons are *congruent.*

If two polygons have their corresponding parts equal and arranged in reverse order, the polygons are said to be *symmetrical.*

In general, two symmetrical spherical polygons do not coincide when one is applied to the other. For example, if the symmetrical spherical triangles ABC and ABD are placed so as to have the side AB in common, the vertices will fall on opposite sides of AB. If the triangle ABD is turned over so that D and C are on the same side of AB, the convex surfaces will face each other and the triangles will not coincide.

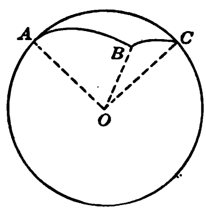


Two spherical polygons are said to be *vertical* when the vertices of one are at the opposite extremities of the diameters drawn from the vertices of the other.

Two spherical pyramids are said to be *symmetrical* when their bases are symmetrical polygons.

Proposition 322 Theorem

The sum of any two sides of a spherical triangle is greater than the third side.



Hypothesis. ABC is a spherical $\triangle$ on the sphere whose centre is O, and $\widehat{AC}$ is the longest side.

Conclusion. $\widehat{AB} + \widehat{BC} > \widehat{AC}$.

Proof. Draw the radii OA, OB, and OC.

$$\angle AOB \stackrel{\propto}{=} \widehat{AB},\ \angle AOC \stackrel{\propto}{=} \widehat{AC},\ \text{and}\ \angle BOC \stackrel{\propto}{=} \widehat{BC}. \quad (?)$$

$$\angle AOB + \angle BOC > \angle AOC. \quad (?)$$

$$\therefore \widehat{AB} + \widehat{BC} > \widehat{AC}. \quad (?)$$

COR. *The difference between two sides of a spherical triangle is less than the third side.*

Proposition 323 Theorem

The sum of the sides of a convex spherical polygon is less than the circumference of a great circle.

HINT. Draw radii from the centre of the sphere to the vertices of the polygon. Consult Prop. 246 and Prop. 126.

Ex. 1173. One side of a convex spherical polygon is less than the sum of the other sides.

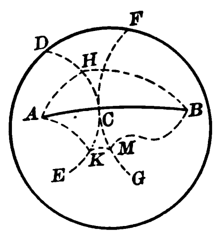
Ex. 1174. If two spherical triangles have a side in common and two of the remaining sides intersect, the sum of the intersecting sides is greater than the sum of the sides which do not intersect.

Ex. 1175. The sum of two sides of a spherical triangle is greater than the sum of arcs of great circles drawn from a point within the triangle to the extremities of the third side.

Ex. 1176. If a point within a spherical triangle is connected with the vertices of the triangle by arcs of great circles, the sum of these arcs is less than the perimeter of the triangle, and is greater than one half the perimeter.

Proposition 324 Theorem

The shortest line that can be drawn on a sphere between two points is the minor arc of a great circle joining the two points.



Hypothesis. AB is a minor arc of a great ⊙ joining the points A and B on a sphere.

Conclusion. AB is the shortest line that can be drawn on the sphere between A and B.

Proof. Let C be any point in $\widehat{AB}$.

With A as a pole and with AC as a polar distance, describe $\widehat{DE}$. With B as a pole and with BC as a polar distance describe $\widehat{FG}$.

Let H be any point other than C in $\widehat{DE}$. Draw great ⊙ arcs AH and BH.

$$\text{Then } AH + BH > AC + BC. \qquad (?)$$

$$\text{But } AH = AC. \qquad (?)$$

$$\therefore BH > BC. \qquad (?)$$

$$\therefore H \text{ is not a point in } \widehat{FG}.$$

Accordingly, C is the only point common to $\widehat{DE}$ and $\widehat{FG}$.

Let $AKMB$ be any line drawn from A to B on the sphere, not passing through C, and cutting $\widehat{DE}$ and $\widehat{FG}$ at K and M respectively.

If, now, the sphere is revolved about the axis drawn through A, the point K will move along $\widehat{ED}$. When K reaches the position C, there will be a line on the sphere drawn from A to C equal to the line AK. Then, whatever the nature of the line AK, an equal line can be drawn from A to C.

In like manner it may be proved that, whatever the nature of the line BM, an equal line can be drawn from B to C.

Hence a line can be drawn through C from A to B equal to the sum of AK and BM, and consequently less than $AKMB$ by the segment KM.

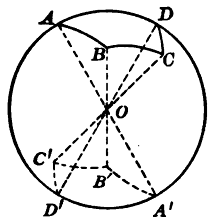
$\therefore$ the shortest line joining A and B must pass through C.

Since C is any point in $\widehat{AB}$, the shortest line joining A and B must pass through every point of $\widehat{AB}$; that is, $\widehat{AB}$ is the shortest line that can be drawn on the sphere between A and B.

Q.E.D.

Proposition 325 Theorem

Two vertical spherical polygons are symmetrical.



Hypothesis. On the sphere ADA', of which O is the centre, polygons $ABCD$ and $A'B'C'D'$ have for their vertices the extremities of the diameters AA', BB', CC', and DD'.

Conclusion. Polygons $ABCD$ and $A'B'C'D'$ are symmetrical.

HINT. To prove that the sides are equal, each to each, consult Prop. 5 and Prop. 100. To prove that the angles are equal consult Prop. 229, Cor. II.

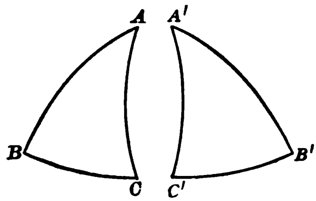
Looking at the two polygons from O, the vertices A, B, C, and D succeed each other in a counter-clockwise order, whereas the vertices A', B', C', and D' succeed each other in a clockwise order; hence the parts are arranged in reverse order.

COR I. *Two symmetrical polygons on a sphere can be placed so as to take the position of vertical polygons.*

COR. II. *The corresponding sides of two vertical polygons are arcs of the same great circle.* This circle is formed by the plane determined by the diameters joining the two pairs of opposite vertices.

Proposition 326. Theorem

Two symmetrical isosceles spherical triangles are congruent.



Hypothesis. $\triangle\!\!\!\triangle\, ABC$ and $A'B'C'$ are two symmetrical spherical $\triangle\!\!\!\triangle$, which in $\angle A = \angle A'$, $AB = A'B'$, and so on. Furthermore, the $\triangle\!\!\!\triangle$ are isosceles; $AB = AC$, and $A'B' = A'C'$.

Conclusion. $\triangle ABC \equiv \triangle A'B'C'$.

Proof. $AB = A'B'$, and $AB = AC$. (?)

$\therefore AC = A'B'$. (?)

In like manner it may be proved that $AB = A'C'$.

Apply the $\triangle ABC$ to the $\triangle A'B'C'$ so that the $\angle A$ shall coincide with the $\angle A'$.

B will fall on C', and C will fall on B'. (?)

Then BC will coincide with $C'B'$. (?)

$\therefore \triangle ABC \equiv \triangle A'B'C'$. (?)

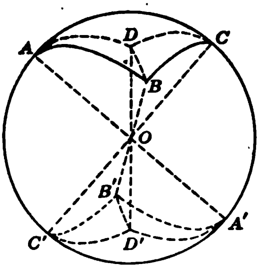
Q. E. D.

COR. I. *If one of two symmetrical spherical triangles is isosceles, the other triangle is isosceles also.*

COR. II. *Triangular spherical pyramids whose bases are symmetrical isosceles triangles are congruent.*

Proposition 327 Theorem

Two symmetrical spherical triangles are equivalent.



Hypothesis. ≙ ABC and $A'B'C'$ are two symmetrical spherical ≙.

Conclusion. $\triangle ABC = \triangle A'B'C'$.

Proof. Place the two ≙ so that they take the position of vertical ≙ with their vertices at the opposite extremities of three diameters.

Let D be the pole of the small ⊙ passing through A, B, and C, and draw great ⊙ arcs DA, DB, and DC.

Then $\overparen{DA} = \overparen{DB} = \overparen{DC}$, and ≙ DAB, DAC, and DBC are isosceles. (?)

Draw the diameter of the sphere DD'. Also draw great ⊙ arcs $D'A'$, $D'B'$, and $D'C'$.

≙ $D'A'B'$, $D'A'C'$, and $D'B'C'$ are isosceles. (?)

$\therefore \triangle DAB \equiv \triangle D'A'B'$, $\triangle DAC \equiv \triangle D'A'C'$,

and $\triangle DBC \equiv \triangle D'B'C'$. (?)

$\therefore \triangle ABC = \triangle A'B'C'$. (?)

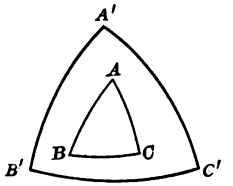
Q. E. D

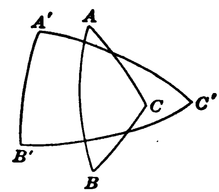
Cor. *Two symmetrical triangular spherical pyramids are equivalent.*

NOTE. In case D and D' are without the triangles ABC and $A'B'C'$, each triangle is equivalent to the sum of two isosceles triangles diminished by a third, and the triangles are equivalent.

POLAR TRIANGLES

If, with the vertices of a spherical triangle as poles, arcs of great circles are drawn, a spherical triangle is formed, which is called the *polar triangle* of the first.





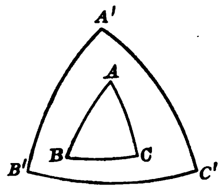
For example, if $B'C'$, $A'C'$, and $A'B'$ are arcs of great circles whose poles are A, B, and C respectively, $\triangle A'B'C'$ is the polar triangle of $\triangle ABC$.

If, instead of the arcs $B'C'$, $A'C'$, and $A'B'$, complete circles are described, the surface of the sphere will be divided into eight spherical triangles. Of these eight triangles, the polar triangle is the one which fulfils the following condition: the corresponding vertices A and A' lie on the same side of BC, the corresponding vertices B and B' lie on the same side of AC, and the corresponding vertices C and C' lie on the same side of AB. The distances AA', BB', and CC' are each less than a quadrant. This test should be applied in proving that one spherical triangle is the polar of another.

NOTE. A spherical triangle may be entirely within or entirely without its polar, or some of the sides of the two triangles may intersect. In some cases a side of one triangle may fall on a side of the other.

Proposition 328 Theorem

If one spherical triangle is the polar triangle of another, then, reciprocally, the second is the polar triangle of the first.



Hypothesis. $\triangle A'B'C'$ is the polar $\triangle$ of $\triangle ABC$.

Conclusion. $\triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$.

Proof. B is the pole of $\overset{\frown}{A'C'}$. (?)

$\therefore A'$ is at a quadrant's distance from B. (?)

Likewise, C is the pole of $\overset{\frown}{A'B'}$, and A' is at a quadrant's distance from C.

$\therefore A'$ is the pole of $\overset{\frown}{BC}$. (?)

In like manner it may be proved that B' is the pole of $\overset{\frown}{AC}$, and that C' is the pole of $\overset{\frown}{AB}$.

Moreover, the distances AA', BB', and CC' are each less than a quadrant.

$\therefore \triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$. (?)

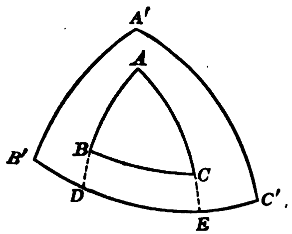
Q. E. D.

NOTE. The proof is identically the same if triangle $A'B'C'$ is within the triangle ABC or if the sides of the two triangles intersect.

DEFINITION. Two spherical triangles, each of which is the polar triangle of the other, are called *polar triangles.*

Proposition 329 Theorem

In two polar triangles, each angle of one is measured by the supplement of the side lying opposite to it in the other.



Hypothesis. $\triangle ABC$ and $A'B'C'$ are polar $\triangle$.

Conclusion. $\angle A \stackrel{\propto}{=} 180° - \overset{\frown}{B'C'}$, $\angle A' \stackrel{\propto}{=} 180° - \overset{\frown}{BC}$, etc.

Proof. Produce, if necessary, $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ to meet $\overset{\frown}{B'C'}$ at D and E respectively.

$\overset{\frown}{B'E}$ and $\overset{\frown}{DC'}$ are quadrants. (?)

$\therefore \overset{\frown}{B'E} + \overset{\frown}{DC'} = 180°.$ (?)

That is, $\overset{\frown}{B'D} + \overset{\frown}{DE} + \overset{\frown}{DC'} = 180°,$

or $\overset{\frown}{B'C'} + \overset{\frown}{DE} = 180°.$

$\angle A \stackrel{\propto}{=} \overset{\frown}{DE}.$ (?)

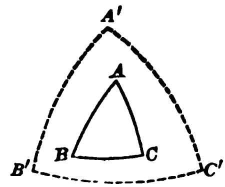
$\therefore \angle A \stackrel{\propto}{=} 180° - \overset{\frown}{B'C'}.$ (?)

By producing $\overset{\frown}{BC}$ to meet $\overset{\frown}{A'B'}$ and $\overset{\frown}{A'C'}$ it can be proved that $\angle A' \stackrel{\propto}{=} 180° - \overset{\frown}{BC}.$

Q.E.D.

Proposition 330 Theorem

The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.



Hypothesis. Let A, B, and C denote the numerical measures of the $\measuredangle$s of the spherical $\triangle ABC$ expressed in degrees.

Conclusion. $180° < A + B + C < 540°$.

Proof. Construct the polar $\triangle A'B'C'$, and let a', b', and c' denote the numerical measures of the sides opposite the $\measuredangle$s A, B, and C respectively.

$A = 180° - a'$, $B = 180° - b'$, and $C = 180° - c'$. (?)

$\therefore A + B + C = 540° - (a' + b' + c')$. (?)

That is, $A + B + C < 540°$.

Now $a' + b' + c' < 360°$. (?)

$\therefore A + B + C > 180°$. (?)

Q. E. D.

Cor. I. *A spherical triangle can have one, two, or three right angles, or one, two, or three obtuse angles.*

Cor. II. *The sum of the angles of a spherical polygon of n sides is greater than $(n - 2)$ 180°.*

Definitions. A spherical triangle having two right angles is called a *bi-rectangular* triangle; a spherical triangle having three right angles is called a *tri-rectangular* triangle.

The excess of the sum of the angles of a spherical triangle over 180° is called its *spherical excess.*

The excess of the sum of the angles of a spherical polygon of n sides over $(n - 2)$ 180° is called its *spherical excess.*

*Proposition 331 Theorem

In a bi-rectangular spherical triangle the sides opposite the right angles are quadrants, and the third side of the measure of the third angle.

Consult Prop. 320, Prop. 234, and Prop. 321.

Cor. I. *If two sides of a spherical triangle are quadrants, the third side is the measure of the opposite angle.*

Cor. II. *Each side of a tri-rectangular spherical triangle is a quadrant.*

Cor. III. *Three planes passing through the centre of a sphere mutually perpendicular to each other divide the sphere into eight congruent tri-rectangular triangles.*

Note. On the earth the triangle formed by two meridians and the equator is an example of Prop. 331. The longitude between two places is the angle formed by the meridians passing through these places, and it is measured by the arc of the equator intercepted between the meridians.

Ex. 1177. If two sides of a spherical triangle are quadrants, the angles opposite these sides are right angles.

Ex. 1178. If three sides of a spherical triangle are quadrants, the triangle is tri-rectangular.

Ex. 1179. The polar triangle of a bi-rectangular spherical triangle is also bi-rectangular.

Ex. 1180. A tri-rectangular spherical triangle coincides with its polar triangle.

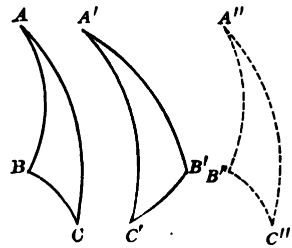
Proposition 332 Theorem

On the same sphere, or on equal spheres, two spherical triangles are either congruent or symmetrical and equivalent when a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other.

CASE I. *When the equal parts are arranged in the same order, the triangles are congruent.*

The method of proof is the same as that of Prop. 9.

CASE II. *When the equal parts are arranged in reverse order, the triangles are symmetrical and equivalent.*



Hypothesis. In the $\triangle$s ABC and $A'B'C'$, $BC = B'C'$, $\angle B = \angle B'$, and $\angle C = \angle C'$, and the parts are arranged in reverse order.

Conclusion. $\triangle$s ABC and $A'B'C'$ are symmetrical and equivalent.

Proof. Let $\triangle A''B''C''$ be symmetrical to $\triangle A'B'C'$.

Then $BC = B''C''$, $\angle B = \angle B''$, and $\angle C = \angle C''$. (?)

$\therefore \triangle ABC \equiv \triangle A''B''C''$. (?)

$\therefore$ $\triangle$s ABC and $A'B'C'$ are symmetrical. (?)

$\therefore \triangle ABC = \triangle A'B'C'$. (?)

Q.E.D.

Proposition 333 Theorem

On the same sphere, or on equal spheres, two spherical triangles are either congruent or symmetrical and equivalent when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

Use a method similar to that used in Prop. 332.

Proposition 334 Theorem

On the same sphere, or on equal spheres, if two spherical triangles are mutually equilateral, they are also mutually equiangular, and are either congruent or symmetrical and equivalent.

HINT. Draw radii from the center of the sphere to the vertices of the ⊿s. Consult Prop. 126, Prop. 244, and Prop. 320.

Proposition 335 Theorem

On the same sphere, or on equal spheres, if two spherical triangles are mutually equiangular, they are also mutually equilateral, and are either congruent or symmetrical and equivalent.

HINT. Construct the polar ⊿s of the given ⊿s, and consult Prop. 329 and Prop. 334.

Ex. 1181. Every point in the great circle perpendicular to a great circle arc at its mid-point is equidistant from the extremities of the arc.

Ex. 1182. State and prove the converse of Ex. 1181.

*Proposition 336 Theorem

In an isosceles spherical triangle, the angles opposite the equal sides are equal.

HINT. Draw great ⊙ arc from the vertex of the △ to the mid-point of the base, and consult Prop. 334.

COR. *The great circle arc drawn from the vertex of an isosceles spherical triangle to the mid-point of base bisects the vertical angle and is perpendicular to the base.*

*Proposition 337 Theorem

If two angles of a spherical triangle are equal, the sides opposite these angles are equal, and the triangle is isosceles.

HINT. Construct the polar △ of the given △, and consult Prop. 329 and Prop. 336.

*Proposition 338 Theorem

If two angles of a spherical triangle are unequal, the side opposite the greater angle is longer than the side opposite the less.

The method of proof is the same as that of Prop. 17.

*Proposition 339 Theorem

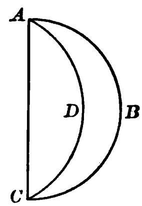
If two sides of a spherical triangle are unequal, the angle opposite the longer side is greater than the angle opposite the shorter.

Use the indirect method.

Ex. 1183. If two adjacent sides of a spherical quadrilateral are longer than the other two sides, the angle included by the two shorter sides is greater than angle included by the two longer sides.

SPHERICAL AREAS AND VOLUMES

A *lune* is a portion of a sphere bounded by two halves of great circles. The angle formed by the semicircles which bound a lune is called the *angle of the lune.* For example, $ABCDA$ is a lune, and BAD is its angle. The lune may be designated by "lune A."



A *spherical wedge,* or *ungula,* is a solid bounded by a lune and the planes of its bounding semicircles. The lune is called the *base* of the wedge, and the diameter in which the planes of the semicircles intersect is called the *edge* of the wedge. The angle of the lune is also called the *angle of the wedge.*

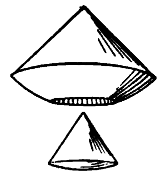
A *zone* is a portion of a sphere included between two parallel planes. The sections made by the parallel planes are called the *bases* of the zone, and the perpendicular distance between the planes is called the *altitude* of the zone.

A *zone of one base* is a zone one of whose bounding planes is tangent to the sphere.

A *spherical segment* is a portion of the volume of a sphere included between two parallel planes. The portions of the planes bounding the segments are called the *bases* of the segment, and the perpendicular distance between the planes is called the *altitude* of the segment.

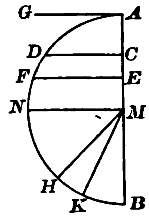
A *spherical segment of one base* is a segment one of whose bounding planes is tangent to the sphere.

NOTE. The curved surface of a segment is a zone.



Spherical Sectors

If a semicircle is revolved about its diameter as an axis, the solid generated by any sector of the semicircle is called a *spherical sector*. The zone generated by the arc of the circular sector is called the *base* of the spherical sector. A spherical sector whose base is a zone of one base is called a *spherical cone*.



As the semicircle AFB revolves about the diameter AB as an axis, the figure $DFEC$ generates a segment of two bases, and the arc DF generates a zone. Likewise, the figure ADC generates a segment of one base, and the arc AD generates a zone. The circular sector MHK generates a spherical sector whose base is generated by the arc HK, and the circular sector MKB generates a spherical cone.

NOTE. If a line AB revolves about another line in its plane as an axis, the area of the surface generated is designated by "area AB." If a closed figure ABC revolves about a line in its plane as an axis, the volume of the solid generated is designated by "vol. ABC."

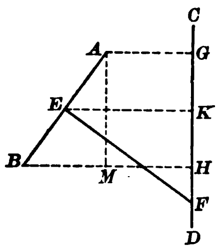
Proposition 340 Theorem

If a straight line revolves about an axis situated in the same plane as the line, but not intersecting it, the area of the surface generated is equal to the product of the projection of the line upon the axis and the circumference of the circle whose radius is the perpendicular to the line erected at its mid-point and terminated by the axis.

CASE I. *When the line is parallel to the axis.*

The surface generated is the lateral surface of a cylinder of revolution. Consult Prop. 287.

CASE II. *When the line is not parallel to the axis and does not meet it.*



Hypothesis. AB and CD are two lines in the same plane which are not $\parallel$ and do not meet, and AB revolves about CD as an axis. GH is the projection of AB on CD, and EF is the $\perp$ to AB erected at its mid-point and terminated by CD.

Conclusion. Area $AB = GH \times 2\pi EF$.

Proof. The surface generated by AB is the lateral surface of a frustum of a cone of revolution.

Draw $EK \perp$ to CD and $AM \perp$ to BH.

$$\triangle ABM \sim \triangle EFK \qquad (?)$$

$$\therefore \frac{AB}{EF} = \frac{AM}{EK} = \frac{GH}{EK}. \qquad (?)$$

$$\therefore \frac{AB}{GH} = \frac{EF}{EK} = \frac{2\pi EF}{2\pi EK}. \qquad (?)$$

$$\therefore AB \times 2\pi EK = GH \times 2\pi EF. \qquad (?)$$

$$\text{Now area } AB = AB \times 2\pi EK. \qquad (?)$$

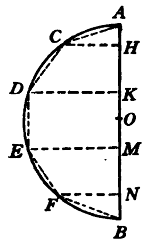
$$\therefore \text{area } AB = GH \times 2\pi EF. \qquad (?)$$

CASE III. *When one extremity of the line lies in the axis.*

The area generated is the lateral surface of a cone of revolution. The method of proof is similar to that of Case II.

Proposition 341 Theorem

The area of a sphere is equal to the product of its diameter and the circumference of a great circle.



Hypothesis. Let a sphere be generated by the revolution of the semicircle AEB, of which O is the centre, about the diameter AB as an axis. Let S denote the area, d the diameter, and r the radius of the sphere.

Conclusion. $S = d \times 2\pi r.$

Proof. Divide the semicircle into any number of equal parts, and draw the chords AC, CD, DE, etc., joining the points of division. The figure thus formed is one half of a regular polygon inscribed in the semicircle. (?)

Let a denote the apothem of this polygon.

Draw CH, DK, EM, etc., $\perp$ to AB.

As the semicircle revolves about AB as an axis,

$$\text{area } AC = AH \times 2\pi a,$$

$$\text{area } CD = HK \times 2\pi a, \text{ etc.} \qquad (?)$$

$$\therefore \text{area } ACDEFB = d \times 2\pi a. \qquad (?)$$

If now, the number of parts into which the semicircle is divided is increased indefinitely,

$$\text{broken line } ACDEFB \doteq \text{semicircle},$$

$$\text{and area } ACDEFB \doteq S. \qquad (?)$$

$$a \doteq r. \qquad (?)$$

The numbers which express the values of the variables area $ACDEFB$ and $d \times 2\pi a$ are one and the same, whatever the number of parts into which the semicircle is divided.

Hence the numbers which express the values of the limits are one and the same, and

$$S = d \times 2\pi r.$$

Q. E. D

COR. I. *The area of a sphere is equal to* 4π *times the square of its radius.*

$$S = 2r \times 2\pi r = 4\pi r^2$$

COR. II. *The area of a sphere is equal to* π *times the square of its diameter.*

$$S = 4\pi r^2 = 4\pi \left(\frac{d}{2}\right)^2 = \pi d^2.$$

COR. III. *The area of a sphere is equal to the area of four great circles.*

COR. IV. *The areas of two spheres are to each other as the squares of their radii, or as the squares of their diameters.*

DISCUSSION. For the purpose of measuring surfaces on a sphere it is convenient to use a unit which is a fractional part of a sphere. A *spherical degree* is $\frac{1}{720}$ of a sphere. As in Plane Geometry a degree is $\frac{1}{90}$ of a quadrant, so in Solid Geometry a spherical degree is $\frac{1}{90}$ of a tri-rectangular triangle. A spherical degree may be conceived of as a bi-rectangular triangle whose third angle is an angle of one degree. Accordingly, *the number of spherical degrees in a bi-rectangular triangle is the same as the number of degrees in the third angle, and the number of spherical degrees in a lune is equal to twice the number of degrees in the angle of the lune.*

It must be distinctly borne in mind that a spherical degree is a surface measure, not an angular measure. The size of a spherical degree depends upon the size of the sphere.

Proposition 342 Theorem

The area of a zone is equal to the product of its altitude and the circumference of a great circle.

Use a method of proof similar to that used in Prop. 341.

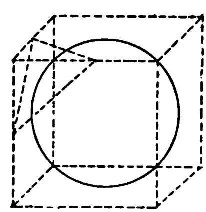
Cor. I. Let Z denote the area of a zone, h the altitude of the zone, and r the radius of the sphere.

$$Z = 2\pi rh.$$

Cor. II. *On the same sphere, or on equal spheres, the areas of two zones are to each other as their altitudes.*

Proposition 343 Theorem

The volume of a sphere is equal to one third the product of its area and its radius.



Hypothesis. Let V denote the volume, S the area, and r the radius of a sphere.

Conclusion. $V = \frac{1}{3} S \times r.$

Proof. Circumscribe a cube about the sphere.

Pass planes through the centre of the sphere and the edges of the cube, thus dividing the cube into six quadrangular pyramids, whose bases are the faces of the cube, and whose common altitude is r.

Let V' denote the sum of the volumes of the pyramids, and S' the sum of their bases.

Then $V' = \frac{1}{3} S' \times r$. (?)

Cut off the corners of the cube by planes tangent to the sphere, thus forming a polyhedron whose volume is nearer the volume of the sphere than is the volume of the cube.

If, now, this process is continued indefinitely, the differences between V' and V and between S' and S can be made so small as to become and remain less than any assigned quantity, however small.

$\therefore V' \doteq V$ and $S' \doteq S$. (?)

The numbers which express the values of the variables V' and $\frac{1}{3} S' \times r$ are one and the same, whatever the number of faces of the polyhedron.

Hence the numbers which express the values of the limits are one and the same, and

$$V = \frac{1}{3} S \times r.$$

Q. E. D.

COR. I. *The volume of a sphere is equal to* $\frac{4}{3}\pi$ *times the cube of its radius.*

$$V = \frac{1}{3}S \times r = \frac{1}{3} \times 4\pi r^2 \times r = \frac{4}{3}\pi r^3.$$

COR. II. *The volume of a sphere is equal to* $\frac{1}{6}\pi$ *times the cube of its diameter.*

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \left(\frac{d}{2}\right)^3 = \frac{1}{6}\pi d^3.$$

COR. III. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

Ex. 1184. The area of a sphere is equal to the lateral area of the circumscribed cylinder of revolution.

Ex. 1185. The area of a sphere is equal to two thirds the total area of the circumscribed cylinder of revolution.

Ex. 1186. The total area of a hemisphere is equal to three fourths the area of the sphere.

Ex. 1187. If a right cylinder is circumscribed about a sphere, the area of a zone on the sphere is equal to the lateral surface of the cylinder included between the planes of the zone.

Ex. 1188. The volume of a sphere is equal to two thirds the volume of the circumscribed cylinder of revolution.

Ex. 1189. The volume of a sphere is equal to twice the volume of a cone whose altitude is equal to the diameter of the sphere and the radius of whose base is equal to the radius of the sphere.

Ex. 1190. If a sphere is inscribed in a right cone, the ratio of the volume of the cone to the volume of the sphere is equal to the ratio of the total surface of the cone to the surface of the sphere.

Proposition 344 Theorem

The volume of a spherical sector is equal to one third the product of the area of the zone which forms its base and the radius of the sphere.

Use a method of proof similar to that used in Prop. 343.

COR. Let V denote the volume of a spherical sector, Z the area of the zone which forms its base, h the altitude of the zone, and r the radius of the sphere.

$$V = \tfrac{1}{3}Z \times r = \tfrac{1}{3} \times 2\pi rh \times r = \tfrac{2}{3}\pi r^2 h.$$

Proposition 345 Theorem

On the same sphere, or on equal spheres, two lunes are equal when their angles are equal.

Use the method of superposition.

COR. *In the same sphere, or in equal spheres, two spherical wedges are equal when their angles are equal.*

Proposition 346 Theorem

On the same sphere, or on equal spheres, two lunes are in the same ratio as their angles.

CASE I. *When the angles of the lune are commensurable.*

CASE II. *When the angles of the lune are incommensurable.*

Use the method of proof given in Prop. 230 and consult Prop. 345.

COR. I. *In the same sphere, or in equal spheres, two spherical wedges are in the same ratio as their angles.*

COR. II. *The ratio of the area of a lune to the area of the sphere is equal to the ratio of the angle of the lune to four right angles.*

Let L denote the area of a lune, A its angle expressed in degrees, and S the area of the sphere.

$$\frac{L}{S} = \frac{A}{360}.$$

COR. III. *The ratio of the volume of a spherical wedge to the volume of the sphere is equal to the ratio of the angle of the spherical wedge to four right angles.*

Let W denote the volume of a spherical wedge, A its angle expressed in degrees, and V the volume of the sphere.

$$\frac{W}{V} = \frac{A}{360}.$$

COR. IV. *The volume of a spherical wedge is equal to one third the product of the lune which forms its base and the radius of the sphere.*

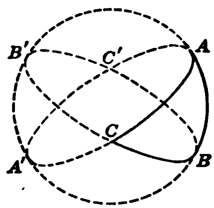
$$W = \frac{A}{360} \times V = \frac{A}{360} \times \frac{1}{3} S r = \frac{1}{3} L r.$$

Ex. 1191. Two lunes on unequal spheres having equal angles are to each other as the squares of the radii of the spheres.

Ex. 1192. If a right cylinder is circumscribed about a sphere, the area of a lune on the sphere is equal to the lateral area of that part of the cylinder intercepted by the planes of the sides of the lune.

Proposition 347 Theorem

The area of a spherical triangle, expressed in spherical degrees, is numerically equal to its spherical excess expressed in degrees.



Hypothesis. Let A, B, and C denote the numerical measures of the ∠s of the spherical △ ABC, and let E denote its spherical excess, all expressed in degrees. Let T denote the area of the △, expressed in spherical degrees.

Conclusion. $T = E.$

Proof. Complete the great ⊙s of which AB, AC, and BC are arcs.

$\triangle ABC + \triangle A'BC =$ lune A. (?)

$\triangle ABC + \triangle AB'C =$ lune B. (?)

$\triangle ABC + \triangle ABC' =$ lune C. (?)

$\triangle A'B'C = \triangle ABC'$. (?)

$\therefore \triangle ABC + \triangle A'B'C =$ lune C. (?)

$$\therefore 3\triangle ABC + \triangle A'BC + \triangle AB'C + \triangle A'B'C$$
$$= \text{lune } A + \text{lune } B + \text{lune } C. \quad (?)$$

But $\triangle ABC + \triangle A'BC + \triangle AB'C + \triangle A'B'C$
$$= \text{hemisphere}. \quad (?)$$
$$\therefore 2\triangle ABC = \text{lune } A + \text{lune } B + \text{lune } C - \text{hemisphere}. \quad (?)$$
$$\therefore 2T = 2A + 2B + 2C - 360°. \quad (?)$$
$$\therefore T = A + B + C - 180° = E. \quad (?)$$

Q. E. D.

COR. I. *The ratio of the area of a spherical triangle to the area of the sphere is equal to the ratio of the spherical excess of the triangle to eight right angles.*

Let T denote the area of a spherical triangle, E its spherical excess, and S the area of the sphere.

$$\frac{T}{S} = \frac{E}{720}.$$

COR. II. *The ratio of the area of a spherical polygon to the area of the sphere is equal to the ratio of the spherical excess of the polygon to eight right angles.*

Let P denote the area of a spherical polygon, E its spherical excess, and S the area of the sphere.

$$\frac{P}{S} = \frac{E}{720}.$$

COR. III. *The ratio of the volume of a spherical pyramid to the volume of the sphere is equal to the ratio of the spherical excess of the base of the pyramid to eight right angles.*

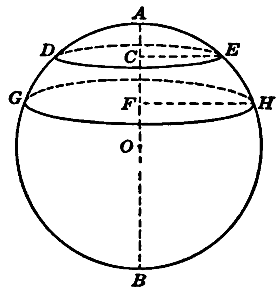
Let Y denote the volume of a spherical pyramid, E the spherical excess of its base, and V the volume of the sphere.

$$\frac{Y}{V} = \frac{E}{720}.$$

COR. IV. *The volume of a spherical pyramid is equal to one third the product of its base and the radius of the sphere.*

Proposition 348 Problem

To compute the volume of a segment of a sphere in terms of the altitude and the radii of the bases.



CASE I. *When the segment is a segment of one base.*

In a sphere with a radius equal to r, let V denote the volume of the spherical segment generated by revolving the figure CEA about the diameter AB as an axis; let r_1 and h denote respectively the radius of the base and the altitude of the segment.

Vol. of spherical sector $O-DAE$

$$= \frac{2}{3}\pi r^2 h. \qquad (?)$$

$$\text{Vol of cone } ODE = \frac{1}{3}\pi r_1^2 (r-h). \qquad (?)$$

$$\therefore V = \frac{2}{3}\pi r^2 h - \frac{1}{3}\pi r_1^2 (r-h)$$

$$= \frac{1}{3}\pi [2r^2 h - r_1^2 (r-h)]. \qquad (?)$$

$$r_1^2 = AC \times CB = h(2r-h). \qquad (?)$$

$$\text{Hence } V = \frac{1}{3}\pi [2r^2 h - h(2r-h)(r-h)]$$

$$= \frac{1}{3}\pi h [2r^2 - 2r^2 + 2rh + rh - h^2]$$

$$= \frac{1}{3}\pi h (3rh - h^2) = \pi h^2 \left(r - \frac{h}{3}\right). \qquad (?)$$

$$\text{Now } r_1^2 = 2rh - h^2, \text{ and } rh = \frac{1}{2}(h^2 + r_1^2). \qquad (?)$$

$$\therefore V = \frac{1}{3}\pi h [\tfrac{3}{2}(h^2 + r_1^2) - h^2]$$

$$= \frac{1}{2}\pi r_1^2 + \frac{1}{6}\pi h^3. \qquad (?)$$

CASE II. *When the segment is a segment of two bases.*

In the sphere with a radius equal to r, let V denote the volume of the spherical segment generated by revolving the figure $FHEC$ about the diameter AB as an axis; let h denote the altitude, and r_1 and r_2 the radii of lower and upper bases respectively. Denote AF by a and AC by b.

$$\text{Vol. of segment } GAH = \pi a^2 \left(r - \frac{a}{3}\right). \quad (?)$$

$$\text{Vol. of segment } DAE = \pi b^2 \left(r - \frac{b}{3}\right). \quad (?)$$

$$\therefore V = \pi r(a^2 - b^2) - \frac{\pi}{3}(a^3 - b^3)$$

$$= \pi (a - b) [r(a + b) - \tfrac{1}{3}(a^2 + ab + b^2)]. \quad (?)$$

$$\text{Now } ra = \tfrac{1}{2}(a^2 + r_1^2) \text{ and } rb = \tfrac{1}{2}(b^2 + r_2^2.). \quad (?)$$

$$\therefore r(a + b) = \tfrac{1}{2}(a^2 + r_1^2) + \tfrac{1}{2}(b^2 + r_2^2). \quad (?)$$

$$\therefore V = \pi h[\tfrac{1}{2}(a^2 + r_1^2) + \tfrac{1}{2}(b^2 + r_2^2) - \tfrac{1}{3}(a^2 + ab + b^2)]$$

$$= \frac{\pi h}{6}[3a^2 + 3r_1^2 + 3b^2 + 3r_2^2 - 2a^2 - 2ab - 2b^2]$$

$$= \frac{\pi h}{6}(3r_1^2 + 3r_2^2 + a^2 - 2ab + b^2)$$

$$= \frac{\pi h}{6}(3r_1^2 + 3r_2^2 + h^2)$$

$$= \tfrac{1}{2}\pi (r_1^2 + r_2^2) h + \tfrac{1}{6}\pi h^3. \quad (?)$$

Q. E. F.

NOTE. If in Case I we wish to find the volume of the segment generated by revolving the figure CEB about the diameter AB as an axis,

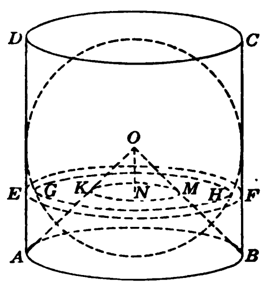
$$V = \tfrac{2}{3}\pi r^2 h + \tfrac{1}{3}\pi r_1^2(h - r)$$
$$= \tfrac{2}{3}\pi r^2 h - \tfrac{1}{3}\pi r_1^2(r - h).$$

REMARK. The formula developed in Prop. 348 may be stated as a theorem as follows:

The volume of a spherical segment is equal to half the sum of its bases multiplied by its altitude increased by the volume of a sphere whose diameter is equal to the altitude of the segment.

Proposition 349 Theorem

If a cylinder of revolution is circumscribed about a sphere and a cone is constructed having a base of the cylinder as its base and the centre of the sphere as its vertex, any section of the sphere made by a plane parallel to the bases of the cylinder is equivalent to the difference between the sections of the cylinder and the cone made by the same plane.



Hypothesis. The cylinder AC is circumscribed about the sphere whose centre is O, and the cone OAB is constructed with $\odot AB$ as a base and O as a vertex. A plane is drawn $\parallel$ to the bases of the cylinder, cutting all three solids and forming $\odot EF$, a section of the cylinder, $\odot GH$, a section of the sphere, and $\odot KM$, a section of the cone.

Conclusion. $\odot GH = \odot EF - \odot KM$.

Proof. Let ON be the $\perp$ distance from O to the plane of the sections, and denote this distance by a. Let the radius of the sphere be denoted by r.

Area of $\odot EF = \pi r^2$. (?)

Radius of $\odot KM = a$,

and area of $\odot KM = \pi a^2$. (?)

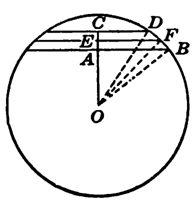
Radius of $\odot GH = \sqrt{r^2 - a^2}$, and area of $\odot GH = \pi (r^2 - a^2)$. (?)

$\therefore \odot GH = \odot EF - \odot KM$. (?)

Q. E. D.

DISCUSSION. If the three solids are cut by two planes parallel to the bases of the cylinder, the spherical segment thus formed is shown by applying Prop. 303 to be equivalent to the difference between the cylinder and the frustum of a cone formed by the same planes. It was shown on page 387 that the volumes of a cylinder and a frustum of a cone can be determined by the prismatoidal formula; hence this formula can be used to determine the volume of a spherical segment.

ILLUSTRATIVE EXAMPLE. The radii of the bases of a spherical segment are 6 in. and 8 in., and its altitude is 2 in. Find the volume.



Let r be the radius, and denote OA by x.

In the $\triangle\ OAB$, $x^2 + 8^2 = r^2$.

In the $\triangle\ OCD$, $(x + 2)^2 + 6^2 = r^2$.

Solving, $x = 6$ and $r = 10$.

In the $\triangle\ OEF$, $(x + 1)^2 + \overline{EF}^2 = 10^2$

$\overline{EF}^2 = 51$

$V = \frac{1}{6}h\,(B + B' + 4M)$

$= \frac{1}{6} \times 2\,(\pi 8^2 + \pi 6^2 + 4\pi \times 51)$

$= \frac{1}{3}\pi\,(64 + 36 + 204) = \frac{304}{3}\pi$

$= 318\frac{10}{21}$ cu. in.

Ex. 1193. By making the proper substitutions in the prismatoidal formula, derive the formula for the volume of a sphere, as given on page 431.

Ex. 1194. By making the proper substitutions in the prismatoidal formula, derive the formula for a spherical segment, as given on page 437.

EXERCISES

Ex. 1195. The shortest line that can be drawn on a sphere from a given point to a given arc of a great circle is an arc of a great circle perpendicular to the given arc.

Ex. 1196. Any point in an arc of a great circle which bisects a spherical angle is equally distant from the sides of the angle.

Ex. 1197. State and prove the converse of Ex. 1196.

Ex. 1198. All the points in a circle of a sphere are at the same distance from any point in its axis.

Ex. 1199. The poles of any two great circles of a sphere lie on a third great circle whose poles are the points of intersection of the given circles.

Ex. 1200. If the poles of two great circles of a sphere lie on a third great circle, the points of intersection of the two circles are the poles of the third circle.

Ex. 1201. The arcs of great circles bisecting the angles of a spherical triangle intersect at a common point which is equidistant from the sides of the triangle.

Ex. 1202. In a spherical triangle the arcs of great circles drawn from the vertices perpendicular to the opposite sides are concurrent.

Ex. 1203. The spherical distance between the poles of two circles of a sphere is a measure of the plane angle of the dihedral angle formed by the planes of the circles.

Ex. 1204. If one side of a spherical angle is a quadrant, an adjacent angle is acute, right, or obtuse, according as the side opposite to it is less than, equal to, or greater than a quadrant.

Ex. 1205. If two spheres are concentric, a plane tangent to the inner sphere touches it at the centre of the circle cut from the outer.

Ex. 1206. A radius of a sphere S is the diameter of a sphere S'. Prove that every chord of S through the point of contact is bisected by the second intersection with S'.

Ex. 1207. Two opposite angles A and C of a spherical quadrilateral are equal, and AB and CB are produced through B to meet the opposite sides produced at E and F respectively. If angles E and F are equal, prove that $AB = BC$.

Ex. 1208. In a plane right triangle the median drawn to the mid-point of the hypotenuse is equal to one half the hypotenuse. Prove that this can never be the case for a right spherical triangle.

Ex. 1209. If the bisector of an angle of a spherical triangle is a quadrant, it bisects the opposite side.

Ex. 1210. If two circles in different planes have two points in common, a sphere can be passed through them.

Ex. 1211. If the plane of a small circle of a sphere is perpendicular to the plane of a great circle, the tangents to the two circles at a point of intersection are mutually perpendicular.

Ex. 1212. If two intersecting planes are tangent to a sphere, the plane determined by the centre and the points of contact is perpendicular to the line of intersection of the two planes.

Ex. 1213. Two planes which intersect in the line AB touch the sphere at the points C and D. Prove that the line CD is perpendicular to AB.

Ex. 1214. If in a spherical triangle the sum of two sides is half a great circle, the sum of the angles opposite these sides is 180°. [HINT — Produce the two sides until they intersect.]

Ex. 1215. A sphere and a right circular cylinder whose axis passes through the centre of the sphere intersect so that each element of the cylinder is cut by the sphere in two points. Prove that the intersection of the sphere and the cylindrical surface consists of two equal circles whose planes are perpendicular to the axis of the cylinder.

Ex. 1216. In a spherical polygon each side is produced through one vertex. Show that the amount by which the sum of the exterior angles so formed falls short of four right angles gives a measure of the area of the polygon.

Ex. 1217. If two spherical triangles have equal perimeters, their polar triangles have equal areas.

Ex. 1218. A zone of one base is equivalent to a circle whose radius is the chord of its generating arc.

Ex. 1219. The volumes of two polyhedrons circumscribed about the same sphere are in the same ratio as the areas of their surfaces.

Ex. 1220. If the north pole of the earth is connected with the equator and the lines are produced until they meet the tangent plane at the south pole, the volume of the cone thus formed is twice the volume of the earth.

Ex. 1221. A square and an equilateral triangle are inscribed in a circle in such a manner that the altitude of the triangle is parallel to a side of the square. These figures all revolve about the altitude of the triangle as an axis, forming a cylinder, a cone, and a sphere. Prove that the total area of the cylinder is a mean proportional between the total area of the cone and the area of the sphere. Prove that the volume of the cylinder is a mean proportional between the volume of the cone and the volume of the sphere.

Ex. 1222. A square and an equilateral triangle are circumscribed about a circle in such a manner that the altitude of the triangle is parallel to a side of the square. These figures all revolve about the altitude of the triangle as an axis, forming a cylinder, a cone, and a sphere. Prove that the total area of the cylinder is a mean proportional between the total area of the cone and the area of the sphere. Prove that the volume of the cylinder is a mean proportional between the volume of the cone and the volume of the sphere.

Ex. 1223. Find the locus of points on a sphere whose spherical distances from two intersecting great circles are equal.

Ex. 1224. Find the locus of points on a sphere which are equidistant from two points within the sphere.

Ex. 1225. Find the locus of a point which is at a distance a from a given plane and at the same time is at a distance $2a$ from a given point in the plane.

Ex. 1226. A point moves so as always to be at a constant distance a from a fixed point A, and at a constant distance b from a fixed plane MN. Determine its locus.

Ex. 1227. Find the locus of the centres of circles cut out of a sphere by planes passing through a point within the sphere.

Ex. 1228. A straight line moves so that it always passes through a fixed point and is tangent to a given sphere. Find the locus of (i) the point of contact; (ii) the moving line.

Ex. 1229. Find the locus of the centre of a sphere tangent to three fixed planes which are mutually perpendicular.

Ex. 1230. Find the locus of points from which three mutually perpendicular planes can be drawn tangent to a sphere whose radius is r.

Ex. 1231. Find the locus of the vertex of a right triangle the sides of which always pass through two fixed points.

PROBLEMS OF COMPUTATION

[Use for π the approximate value $3\frac{1}{7}$]

1. The radius of a sphere is 2 ft. Find the area and the volume of the sphere.

2. The volume of a sphere is 22 cu. ft. Find the radius and the area of the sphere.

3. If a cubic foot of lead weighs 712 lb., find the weight of a ball 3 in. in diameter.

4. Find the diameter of the mouth of a cannon that carries a 24 pound ball when 1 cu. in. of iron weighs $4\frac{1}{2}$ oz.

5. The area of a sphere is $50\frac{2}{7}$ sq. in. Find the volume.

6. Find the diameter of a sphere made by melting together three spheres whose diameters are 12 in., 8 in., and 4 in.

7. A piece of lead 5 in. $\times$ 3 in. $\times$ 6 in. is made into spherical balls, each of which is $\frac{3}{4}$ in. in diameter. Find the number of balls.

8. A cubical box, one foot on a side, is filled with spherical bullets $\frac{1}{2}$ in. in diameter, the bullets being arranged in tiers 24 on a side. Compare the weight of the bullets with the weight of a lead ball 1 ft. in diameter. Compare the total area of the bullets with the area of the ball.

9. The radii of two spheres differ by 3 in., and their areas differ by 264 sq. in. Find their volumes.

10. Find the area of a plane section of a sphere of radius 10 in. which passes 6 in. from the centre.

11. How far from the centre of a sphere of radius r must a plane be passed to cut out a circle whose area is one eighth the area of the sphere?

12. Find a formula for the weight of a spherical shell, the inside radius being r, the thickness of the metal being t, and the weight of a cubic unit of the metal being w.

13. How many cubic inches of rubber will be required to manufacture one thousand hollow rubber balls, having an outside diameter of 2 in., if the rubber is $\frac{1}{8}$ in. thick?

14. A solid glass ball 6 in. in diameter is expanded by a glass blower so that it becomes a hollow sphere 1 in. thick. Find the outer diameter of this hollow sphere.

15. The exterior diameter of a spherical shell is 7 in., and its weight is one tenth that of a solid ball made of the same material and having the same diameter. Find the thickness of the shell.

16. Find the volume of the earth's atmosphere if it extends 50 mi. from the surface, assuming the earth to be a sphere with a radius of 4000 mi.

17. The number of square feet in the area of a sphere is equal to the number of cubic feet in its volume. Find its radius.

18. What is the ratio of the volumes of two spheres if the area of one is double the area of the other?

19. The volumes of two given spheres are in the ratio 1 : 2. Find the ratio of the total areas of their inscribed cubes.

20. If a sphere fits exactly into a cubical box, the volume of the sphere is what fractional part of the volume of the box?

21. What per cent of the volume of a sphere is contained in the inscribed cube?

22. Find the volume of a sphere circumscribing a cube whose volume is 27 cu. ft.

23. Find the area and the volume of a sphere inscribed in a cube whose diagonal is $6\sqrt{3}$ in.

24. A sphere of radius r is placed in the corner of a room touching the floor and both the walls. Find the radius of the largest sphere that will stand in the space behind it.

25. The inner edge of a hemispherical bowl is $\sqrt{6}$ in. Find the edge of the largest cube that can be placed under the bowl.

26. A regular quadrangular pyramid is inscribed in a sphere 10 in. in diameter, and its altitude is 8 in. Find the volume and total area of the pyramid.

27. If a is the length of the edge of a regular tetrahedron, find the volume of the circumscribed sphere.

28. Find the volume of a sphere inscribed in a regular tetrahedron whose edge is a.

29. A regular octahedron has an edge a. Find the volume of the inscribed sphere.

30. Find the edge of a regular octahedron if the radius of the circumscribed sphere is r.

31. A polyhedron is circumscribed about a sphere whose radius is 2 in. If the volume of the polyhedron is 60 cu. in., what is the area of its surface.?

32. The length of a perfectly round log of wood is 20 ft., and the diameter of each end is 12 ft. Find the volume of the greatest sphere which can be cut out of it.

33. The area of a sphere is $452\frac{4}{7}$ sq. in. Find its volume, and the total area and the volume of the circumscribed right cylinder.

34. An iron sphere 7 in. in diameter is in a cylindrical pail 14 in. in diameter and is just submerged in water. How high will the water stand in the pail if the sphere is removed?

35. One hundred spherical bullets $\frac{1}{2}$ in. in diameter are placed in a cylindrical tomato can 4 in. in diameter and 5 in. high. How much water must then be poured in in order to fill the can?

36. A solid has the shape of a cylinder with a hemispherical end, and the diameter and length of the cylindrical portion are each 6 in. Find the total area and volume of the solid.

37. A wooden solid consists of two hemispherical caps fitting exactly on the bases of a cylinder. The length of the cylindrical portion is one half the total length of the solid. Express the total area and the volume in terms of the total length of the solid.

38. A cylindrical box is to be made which will just hold 7 pills. Which is the cheaper shape for it, gun-barrel or grindstone? If the pills are $\frac{1}{4}$ in. in diameter, find the area of the interior of the box.

39. A sphere whose radius is 5 in. is cut down to the form of a cylinder of revolution which could be exactly inscribed in the sphere and have an altitude of 8 in. Find the volume of the part cut away.

40. The altitude of a cone of revolution is 3 in., and the radius of its base is 6 in. What is the radius of a sphere having the same volume?

41. Find the total area of a cone of revolution whose altitude is double the diameter of its base, and whose volume is equal to that of a sphere whose surface contains 154 sq. in.

42. A hemisphere and a right circular cone have the same base and the areas of their curved surfaces are equal. Find the ratio of their volumes.

43. The lateral area of a cone of revolution and the area of a sphere are each equal to 49 sq. ft. If the radius of the sphere equals the radius of the base of the cone, find the altitude of the cone.

44. An iron post has the form of a right circular cylinder 8 in. in diameter and $3\frac{1}{2}$ ft. tall, surmounted by a right circular cone, the diameter of the base of which is 8 in. and the altitude of which is 2 in. The post is melted and cast into the form of a sphere. Find the radius of the sphere.

45. The area of a circle is $113\frac{1}{7}$ sq. in. Find the volume of the sphere generated by revolving it about a diameter as an axis; also find the volume of the cone generated in the same motion by the inscribed equilateral triangle whose base is perpendicular to this diameter.

46. An equilateral triangle revolves about one of its altitudes as an axis. If its altitude is $3h$, find the volumes of the solids generated by the triangle, the inscribed circle, and the circumscribed circle.

47. The radii of a cylinder of revolution and a cone of revolution are the same as the radius r of a sphere. The altitudes of the cylinder and cone are $2r$. Compare the volumes of the cylinder, cone, and sphere.

48. A cylinder of revolution and a cone of revolution each have as base a great circle of a sphere, and as altitude a diameter of the sphere. Find the ratios of the volumes of the cylinder and cone to the volume of the sphere.

49. The diameter of a cone of revolution is equal to its slant height. A sphere is inscribed in the cone, and a cylinder of revolution is circumscribed about the sphere. Find the ratios of the three volumes.

50. Compare the volume of a right prism $6a$ ft. high, the base of which is a square with a side $2a$ ft. long, with the volumes of the largest cylinder, cone, and sphere which can be made from it.

51. From a hemisphere whose radius is 3 in. a solid is cut by means of a cone of revolution whose base coincides with the base of the hemisphere and whose altitude is equal to the radius of the sphere. Find the volume and total area of the solid that is left.

52. The radius of a sphere is 5 ft.; upon a section of the sphere at a distance 3 ft. from the centre as a base a cone is constructed whose elements touch the sphere. Find the volume of the cone.

53. A point is distant from a sphere of radius r by an amount equal to the radius. Find the volume of the cone bounded by the tangents from this point to the sphere and the plane of the circle of contact.

54. The inside of a glass is in the form of a right circular cone whose vertical angle is 60° and whose base is 2 in. across. The glass is filled with water and the largest sphere that can be immersed is placed in the glass. How much water remains in the glass.

55. A conical vessel has the form of a right circular cone of radius 3 in. and depth 4 in. The vessel is filled with water and a marble is dropped in. Find the radius of the marble if the top of the marble is level with the top of the vessel. What per cent of the water remains?

56. A sphere of radius 5 ft. and a right circular cone with a base of radius 5 ft. stand on a horizontal plane. If the height of the cone is equal to the diameter of the sphere, find the position of the horizontal plane which cuts the two solids in equal circular sections. Find also the area of these sections.

57. If the radius of a sphere is 30 ft., find the area of a zone whose altitude is 3 ft.

58. The altitude of the torrid zone is 3200 mi. Find its area. (Radius of the earth = 4000 mi.)

59. On a sphere of radius r find the altitude of a zone whose area is equal to the area of a great circle.

60. The surface of a zone whose altitude is 14 in. contains 2198 sq. in. Find the radius of the sphere.

61. What fractional part of the earth's surface lies north of the parallel of 60° north latitude? Does twice as much lie north of the parallel of 30° north latitude?

62. What fractional part of the earth's surface lies between the equator and the parallel of 45° north latitude?

63. What fractional part of the earth's surface lies between the parallels of 60° north latitude and 45° south latitude?

64. A sphere 4 in. in diameter is bored through the centre by a two inch auger. What is the total area of the remaining solid?

65. A hemispherical dome 150 ft. in diameter is divided into two parts by a horizontal plane which lies seven eighths of the way from the base to the summit. Find the expense of gilding the lower part at a cost of 25 cts. per square foot.

66. Two spheres have radii of 5 in. and 12 in., and their centres are 13 in. apart. Find the area of that portion of each sphere which is outside of the other.

67. The internal and external radii of a spherical shell are respectively 6 in. and $6\frac{1}{2}$ in. A plane, whose distance from the centre of the shell is 5 in., cuts off a piece of the shell. Find the area of the surface of this piece.

68. A certain dome is in the form of a zone of one base. The base is a circle of 24 ft. in diameter, and the highest point is 9 ft. above the plane of the base. Find the number of square feet in the surface of the dome.

69. A zone of one base has an altitude of 4 in., and the diameter of the base is 16 in. Find the radius of the sphere and the area of the zone.

70. The area of a zone on a sphere of radius r is equal to the area of a sphere whose radius is equal to the altitude of the zone. Find the altitude.

71. The total surface of a cube circumscribed about a sphere is 96 sq. ft. What is the area of a zone on the same sphere included by a plane passing through the centre and a parallel plane at a distance of 1 ft. from the centre?

72. On a sphere whose diameter is 14 in., the altitude of a zone of one base is 2 in. Find the altitude of a cylinder having an equal base and curved area.

73. A sphere will just fit inside a cone of revolution whose altitude is h while the radius of the base is r. Find the area of that zone of the sphere which is toward the vertex of the cone.

74. On a sphere whose diameter is 10 ft. find the area of a zone, the diameters of whose upper and lower bases are 6 ft. and 8 ft. respectively.

75. The eight vertices of a cube all lie on a sphere. If one edge of the cube is a, find the area of a zone of one base cut off by the plane of one face of the cube.

76. On a sphere of radius r the area of a certain zone of one base is equal to the lateral area of the cone whose base is the base of the zone and whose vertex is the centre of the sphere. Find the altitude of the zone.

77. A cone of revolution is tangent to a sphere of radius r along a small circle. The area of the curved surface of the cone is equal to the area of that part of the sphere not included within the cone. Compute the distance from the centre of the sphere to the plane of the small circle.

78. The area of a certain small circle on a sphere of radius r is equal to the difference of the areas of the zones into which the circle divides the sphere. Find the distance from the centre of the sphere to the plane of the small circle. If a cone is tangent to the sphere along the given circle, find the distance from the centre of the sphere to the vertex of the cone.

79. Two parallel planes, equidistant from the centre of a sphere of radius r, cut from the sphere a zone whose area is five fourths the area of the curved surface of the cylinder having the same bases as the zone. Find the distance of the planes from the centre of the sphere.

80. Two parallel planes on the same side of the centre of radius r bound a zone. The area of this zone is one fourth that of the sphere. The area of the circle cut by the plane nearer the centre is double that cut by the farther plane. Find the distance from the centre of the sphere to the nearer plane.

81. How far from the eye must a sphere of 10 in. radius be held in order that just one third of its surface may be visible?

82. How high must an aeronaut ascend in order to see one thousandth of the earth's surface? (Radius of earth = 4000 mi.)

83. How high must a man ascend in a balloon over the north pole in order to see all of the earth's surface north of the parallel of 45° north latitude? (Radius of earth = 4000 mi.)

84. If a man were at a diameter's distance above the surface of the earth, what part of its surface could he see?

85. A man in a balloon is 10 mi. above the earth. How many square miles of its surface can he see? (Radius of earth = 4000 mi.)

86. If the diameter of a sphere is 12 ft., find the area of a lune whose angle is 80°.

87. The area of a lune whose angle is 30° is $37\frac{5}{7}$ sq. ft. Find the volume of the sphere.

88. Find the angle of a lune whose area is equal to the area of a great circle.

89. Find the number of degrees in the angle of a lune on a sphere of radius 10 in. such that the area of the lune shall be equal to the total area of a right circular cylinder circumscribed about a sphere of radius 5 in.

90. If the radius of a sphere is 4 ft., find the volume of a spherical wedge whose angle is 45°.

91. Find the angle of a spherical wedge if its volume is one sixth the volume of the sphere.

92. A spherical triangle has angles of 75°, 94°, and 91°. What is its area in spherical degrees? How large a portion of the surface of the sphere does it cover?

93. The angles of a spherical triangle on the surface of a sphere of 14 in. radius are 123°, 60°, and 87°. Find the area of the triangle in square feet.

94. The angles of a spherical triangle are 120°, 145°, and 95°. Find its area if the diameter of the sphere is 7 ft.

95. How many spherical triangles will three great circles make on a sphere? If the angles of one triangle are 60°, 80°, and 100°, while the radius of the sphere is 7 in., find the area of each triangle.

96. Find the angles of an equilateral spherical triangle whose area is equal to the area of a great circle.

97. The area of a spherical triangle is one tenth the area of the surface of the sphere on which it lies. Two angles of the triangle are 96° and 87°. Find the third angle.

98. The sides of a spherical triangle are 30°, 40°, and 50°. Find the area of its polar triangle if the radius of the sphere is 7 in.

99. Each side of an equilateral spherical triangle is 100°. Find the area of its polar triangle if the radius of the sphere is 10 in.

100. The sides of a spherical triangle are 220 in., 165 in., and 132 in., and the radius of the sphere is 105 in. Find the area of the polar triangle.

101. On a sphere of radius 2 ft. the area of a certain spherical triangle is 18 sq. ft. Find the perimeter of the polar triangle.

102. On a sphere with an area of 72 sq. in., find the area of a spherical quadrilateral whose angles are 100°, 120°, 140°, and 160°.

103. Each of the angles of a spherical quadrilateral on the earth's surface is 120°. If the radius of the earth is 4000 mi., what is the area of the quadrilateral?

104. The angles of a spherical pentagon are 90°, 100°, 120°, 140°, and 150°, and the radius of the sphere is 7 in. Find the area of the pentagon.

105. An equiangular spherical hexagon on a sphere of radius 5 ft. has an area of 9π sq. ft. Find the angles of the hexagon.

106. The volume of a sphere is 80 cu. ft. Find the volume of a triangular spherical pyramid, the angles of whose base are 103°, 112°, and 127°.

107. If the radius of a sphere is 6 in., what is the volume of a triangular spherical pyramid, the angles of whose base are 84°, 93°, and 108°?

108. Find the volume of a quadrangular spherical pyramid, the angles of whose base are 107°, 118°, 134°, and 146°, the diameter of the sphere being 12 in.

109. The dihedral angles of a trihedral angle are 100°, 65°, and 87°. The trihedral angle is closed by a portion of a sphere whose radius is 4 in. and whose centre is at the vertex of the trihedral angle. Find the area of this portion of the sphere.

110. The perimeter of a certain spherical triangle is equal to one half of a great circle of the sphere on which it lies. What part of the area of the sphere is the area of the polar triangle? What part of the volume of the sphere is cut out by the planes through the centre of the sphere and the sides of the second triangle?

111. The angles of a spherical triangle on a sphere of radius r are 86°, 113°, and 125°. Find the altitude of a zone on this sphere which has the same area as the triangle.

112. On the same sphere are an equilateral spherical triangle, each of whose angles are 93°, and a lune whose angle is 75°. Find the ratio of the areas of these two figures.

113. What part of the volume of a sphere is the volume of a spherical segment of one base whose altitude is one-half the radius of the sphere.

114. A spherical segment of one base is cut from a sphere of radius r, and the radius of the base of the segment is a. Find the volume of the segment.

115. If the radius of a sphere is 12 ft., find the volume of the spherical sector whose base is a zone with an altitude of 2 ft.

116. A sphere floats in water so that three fourths of its surface is submerged. What part of its volume is submerged?

117. How many gallons of water must be poured into a hemispherical bowl of radius 14 in. in order that the water may be 14 in. deep?

118. The distance of a plane from the centre of a sphere is one third the radius of the sphere. Find the ratio of the volumes of the two solids into which the sphere is divided by this plane.

119. What parallel of latitude possesses the property that one fourth of the earth's surface lies to the north of it? What part of the earth's volume is north of the plane of this parallel?

120. The radius of a sphere is 7 ft. The volume of a segment of one base is one half the volume of the spherical sector of which it forms a part. Find its volume.

121. The volume of a segment of one base of a sphere of radius r is equal to the volume of a sphere whose radius is equal to the altitude of the segment. Find the altitude.

122. Find the volume of a spherical segment if the diameter of each base is 8 ft. and the altitude of the segment is 6 ft.

123. A 90° segment of a circle of radius r is revolved about a line through the centre parallel to its chord. Find the volume thus generated.

124. What part of the volume of the earth is the volume of the segment included between the plane of the equator and the plane of the parallel of 30° north latitude?

125. A sphere of radius 13 in. has a cylindrical hole bored through it, the axis of the cylinder passing through the centre of the sphere and the radius of the cylinder being 5 in. Find the total area and the volume of the solid which is left.

126. The surface of a sphere is $31\frac{3}{7}$ sq. ft. Find the surface of another sphere having three times the volume of the former.

127. The radius of a sphere is r. Find the radius of a sphere whose area is twice that of the given sphere.

128. The radius of a sphere is r. Find the radius of a sphere whose volume is twice that of the given sphere.

129. Find the ratio of the areas of two mutually equiangular spherical triangles, one on a sphere of radius 1 ft., the other on a sphere of radius 2 ft.

FORMULAS OF SOLID GEOMETRY

NOTATION

A	angle of lune	p	perimeter of right section
B, B'	areas of bases	p, p'	perimeters of bases
E	spherical excess	r	radius
e	lateral edge	S	lateral area
h	altitude	S	area of sphere
L	area of lune	T	area of spherical triangle
l	slant height	V	volume
M	area of mid-section	W	volume of spherical wedge
P	area of spherical polygon	Y	volume of spherical pyramid

AREAS

Prism	$S = p \times e$
Regular Pyramid	$S = \frac{1}{2}p \times l$
Frustum of a Regular Pyramid	$S = \frac{1}{2}(p + p') \times l$
Cylinder of Revolution	$S = 2\pi r h$
Cone of Revolution	$S = \pi r l$
Frustum of a Cone of Revolution	$S = \pi (r + r') l$

Sphere	$S = 4\pi r^2$
Zone	$S = 2\pi r h$
Lune	$\frac{L}{S} = \frac{A}{360}$
Spherical Triangle	$\frac{T}{S} = \frac{E}{720}$
Spherical Polygon	$\frac{P}{S} = \frac{E}{720}$

VOLUMES

Prism.	$V = B \times h$
Pyramid	$V = \frac{1}{3}B \times h$
Frustum of a Pyramid . .	$V = \frac{1}{3}h\,(B + B' + \sqrt{B\,B'})$ or *Prismatoidal Formula*
Prismatoid	$V = \frac{1}{6}h\,(B + B' + 4\,M)$
Circular Cylinder	$V = \pi r^2 h$
Circular Cone	$V = \frac{1}{3}\pi r^2 h$
Frustum of a Circular Cone	$V = \frac{1}{3}\pi h\,(r^2 + r'^2 + r\,r')$ or *Prismatoidal Formula*
Sphere	$V = \frac{4}{3}\pi r^3$
Spherical Sector	$V = \frac{2}{3}\pi r^2 h$
Spherical Segment . .	$V = \frac{1}{2}\pi\,(r_1^2 + r_2^2)\,h + \frac{1}{6}\pi h^3$ or *Prismatoidal Formula*
Spherical Wedge	$\frac{W}{V} = \frac{A}{360}$
Spherical Pyramid	$\frac{Y}{V} = \frac{E}{720}$

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Solid Geometry